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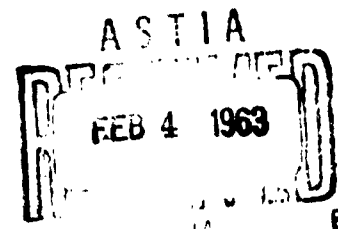
MEMORANDUM  
RM-3485-PR  
JANUARY 1963

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SKIN FRICTION AND STABILITY OF  
A LAMINAR BINARY BOUNDARY LAYER ON  
A FLAT PLATE

J. F. Gross



PREPARED FOR:

UNITED STATES AIR FORCE PROJECT RAND

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The **RAND** Corporation  
SANTA MONICA • CALIFORNIA

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PREFACE

This Memorandum describes one of the first Project RAND studies in a continuing research program on heat and mass transfer in boundary-layer flows. This work formed the basis for subsequent, more complete studies on the problem of re-entry heating, which were summarized in RM-2516, A Review of Binary Boundary Layer Characteristics, by J. F. Gross, J. P. Hartnett, D. J. Masson, and C. Gazley, Jr. Although the work was done in 1956, major portions of the analysis still represent a unique contribution.

The results reported herein were first presented (orally by C. Gazley, Jr.) at the Mass-Transfer Cooling Symposium held at The RAND Corporation in June 1957.

SUMMARY

This Memorandum reviews the problem of a laminar boundary layer on a flat plate with mass transfer. Previous work is reviewed and the fundamental equations are derived. The case of an incompressible laminar boundary layer without heat transfer with hydrogen, carbon dioxide, and iodine injection is solved. Velocity and concentration profiles and skin-friction coefficients are calculated and discussed. It is shown that surface injection of a foreign material into the boundary layer reduces the skin-friction coefficient. The stability of an incompressible laminar boundary-layer with mass transfer is estimated using the Lin approximation and is shown to decrease as the molecular weight of the injected substance decreases.

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LIST OF SYMBOLS

- $A_{ij}$  = coefficients in differential-equation system  
 $c_f Re_{x_e}$  = local skin friction parameter  
 $c_{1w}$  = mass concentration of diffusing component at the wall  
 $c_i$  = mass concentration of  $i^{th}$  component  
 $c_{p_i}$  = specific heat of  $i^{th}$  component  
 $D_{ij}$  = molecular-diffusion coefficient  
 $D'_{jT}, D_{iT}$  = thermal-diffusion coefficients  
 $f$  = dimensionless stream function  
 $g_i$  = chemical potential of  $i^{th}$  component  
 $h_i$  = enthalpy of  $i^{th}$  component  
 $\bar{J}_i$  = diffusion mass velocity of  $i^{th}$  component  
 $k$  = volume viscosity  
 $k_i^T$  = thermal conductivity  
 $L$  = characteristic length  
 $m_i$  = molecular weight of  $i^{th}$  component  
 $n_i$  = molecule concentration of  $i^{th}$  component  
 $P$  = pressure  
 $Pr_e = c_{p_e} \mu_e / k_e$ , Prandtl number  
 $\vec{Q}$  = heat flux vector  
 $R$  = gas constant  
 $Re_L = L \mu_e \rho_e / \mu_e$ , Reynolds number  
 $Sc = \mu / \rho D_{12}$ , Schmidt number  
 $s$  = entropy  
 $T$  = temperature  
 $t$  = time



$u_e$  = velocity at edge of boundary layer  
 $\vec{V}$  = center-of-mass velocity  
 $\vec{V}_i$  = diffusion velocity of  $i^{\text{th}}$  component

$\alpha$  = thermal-diffusion coefficient  
 $\alpha_1$  = wave number of disturbance  
 $\gamma_i$  = average mass velocity of  $i^{\text{th}}$  component  
 $\nabla \cdot \vec{V}$  = the tensor reflexive of  $\nabla \cdot \vec{V}$

$\delta$  = boundary-layer thickness  
 $\eta$  = dimensionless variable =  $\frac{y}{2x} \sqrt{\frac{\rho_\infty u_\infty x}{\mu_\infty}}$   
 $\mu$  = viscosity

$\rho$  = density  
 $\sigma$  = rate of entropy production  
 $\vec{\tau}$  = stress tensor

$\phi_i, \phi_r, \phi'_i, \phi'_r$  = tabulated values given in Lin and Dunn<sup>(5)</sup>  
 $\psi$  = stream function

## I. INTRODUCTION

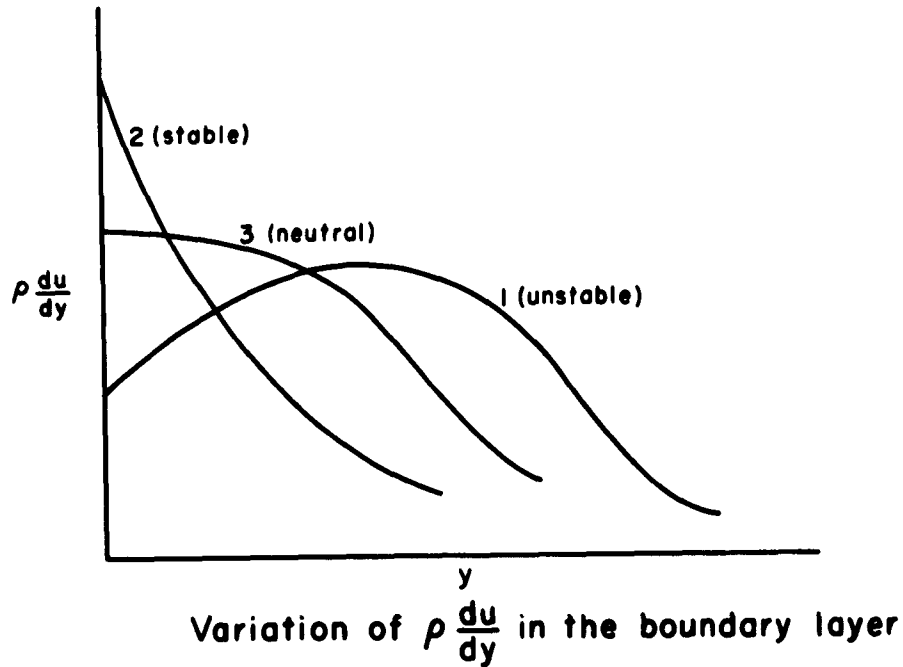
The binary laminar boundary layer may be defined as a laminar boundary layer in which a foreign substance is diffusing. For many years scientists connected with the aircraft industry have been searching for possible methods of controlling the boundary layer on the surfaces of aircraft and missiles. It has been shown experimentally that the friction coefficient for the turbulent boundary layer may be from two to five times as great as that for a laminar layer. Therefore, successful efforts to control the transition point from laminar to turbulent boundary-layer flow would be rewarded by significant savings in power requirements for the vehicle. This has resulted in some critical analyses of the stability of the laminar boundary layer, i.e., the causes for the instability and the conditions under which it occurs.

Several methods for controlling the boundary layer have been proposed. Removing part of the boundary layer through a slotted or porous surface prevents thickening of the boundary layer and consequent turbulence. However, this method also increases the friction coefficient and thus is limited in its applicability. It is also possible to design the airfoil in such a way that transition is delayed as long as possible. However, structural, armor, and fuel requirements play a very important part in the design of an airfoil, and it may be necessary to abandon some conditions in order to fulfill others. Beyond that, the difficulties in aerodynamic design at supersonic speeds are well known. Finally, it has been shown that the boundary layer may be stabilized by cooling. This presents three possibilities: (1) direct removal of heat by conduction to a heat sink, (2) injection into the boundary layer of a lightweight, high-heat-capacity

gas, or (3) use of a solid surface which would sublime or vaporize into the boundary layer at elevated temperatures. The latter two possibilities may involve a binary laminar boundary layer.

The direct removal of heat by transfer through the skin to a heat sink can be shown to be an order of magnitude less efficient than the latter two cooling methods. In the case of extremely high flight speeds, a system employing this method might become incapable of handling the heat loads altogether. Eckert, Schneider, and Köhler<sup>(1)</sup> have shown the beneficial results obtained by injecting a lightweight, high-heat-capacity gas into the boundary layer. They do not take into account the fact that high rates of injection of a lightweight gas act in a destabilizing manner and might bring about early transition to turbulent flow. As pointed out above, this is undesirable from a design point of view. Injecting materials of higher molecular weight into the boundary layer will not give the low friction coefficients that are obtained with hydrogen. However, the delay in transition to turbulent flow may well be the deciding factor here. A qualitative stability analysis will be necessary to indicate the proper cooling method.

A crude method for determining the stability of a compressible laminar flow is to observe the variation of the quantity  $\rho du/dy$  through the boundary layer. If this quantity has a point of inflection for some  $u/u_e > 1 - 1/M_e$  (see Curve 1 of text figure below), the flow is unstable at sufficiently high Reynolds numbers.<sup>(2)</sup> Curve 2 shows the cases for a stable flow and Curve 3 indicates a neutral situation. If, then, the shear remains essentially constant for the addition of small amounts of foreign material to the boundary layer, it might be possible to improve the shape of the  $\rho du/dy$



curve by increasing the density and decreasing the viscosity of the binary boundary layer. It should be noted that this alone will not bring about the desired result, for it is essentially a change of sign, and not magnitude only, that is of interest here. Previous work has shown that the introduction of a foreign, diffusing substance into the boundary layer produces a change in the velocity profile by the addition of a vertical velocity component. The magnitude of this effect will depend on the properties of the injected material. In general, low-density substances will be more destabilizing because they are more effective in reducing the flow acceleration near the wall. This implies the appearance of an inflection point.

Higher-density materials which do not decelerate the flow significantly will as a consequence reduce the skin friction by only a small amount. These two opposite effects must then be considered. A low-density material will cause a large reduction in skin friction but may destabilize the boundary

layer to such an extent that transition will occur. A high-density injection material will not bring about an early transition, but the resulting skin-friction reduction will be minimal. Some compromise will be necessary.

On the basis of the above arguments, it would seem reasonable to search for materials possessing relatively high densities and low viscosities. Two such substances which are feasible from a practical standpoint are carbon dioxide and iodine. Both of these materials sublime, so that it may be possible to employ them without the attendant use of transpiration-cooling plumbing. A qualitative analysis will be employed using the isothermal boundary layer.

## II. FUNDAMENTAL EQUATIONS

The equations governing a laminar binary boundary layer flow will be derived briefly following closely the analysis of Hall.<sup>(3)</sup> In general, we shall be concerned with a multicomponent, nonreacting system. Our further assumptions will be

1. Isotropic medium
2. Single-phase and locally homogeneous medium
3. No external force field
4. Transport phenomena linearly dependent upon property gradients
5. Diffusion velocities small in absolute magnitude
6. Steady state

Since the system is assumed to be nonreacting, the generation term does not appear and the continuity of mass for each component may be expressed as

$$\frac{\partial n_i}{\partial t} + \nabla \cdot n_i (\vec{V} + \vec{V}_i) = 0 \quad (1)$$

where  $n_i$  is the number of molecules of species  $i$  per unit volume and  $\vec{V}_i$  is the diffusion velocity. Summing all the components gives

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} = 0 \quad (2)$$

where

$$\rho = \sum_i m_i n_i = \sum_i \rho_i \quad (3)$$

and

$$\rho \vec{V} = \sum_i m_i n_i (\vec{V} + \vec{V}_i) \quad (4)$$

the last equation defining the center-of-mass velocity  $\bar{V}$ . The diffusion mass velocity is measured by the relative mass motion of a particular component with respect to the center-of-mass velocity:

$$\vec{j}_1 = m_1 n_1 \vec{v}_1 = m_1 n_1 (\vec{v}_1 - \bar{V}) \quad (5)$$

Therefore

$$\vec{v} \cdot \vec{j}_1 = \rho \left[ c_1 \vec{v} \cdot \bar{V} + \bar{V} \cdot \vec{v} c_1 \right] \quad (6)$$

and, finally

$$\rho \frac{dc_1}{dt} + \vec{v} \cdot \vec{j}_1 = 0 \quad (7)$$

Equation (7) is referred to hereafter as the basic diffusion equation.

The equations of motion are simply the well-known Navier-Stokes equations, which have been derived in detail in Ref. 4:

$$\rho \frac{d\vec{V}}{dt} = -\nabla p + \vec{v} \cdot \left( K - \frac{2}{3} \mu \right) \nabla \vec{V} \cdot \vec{V} + \mu (\nabla \vec{V} + \vec{v} \cdot \nabla) \quad (8)$$

It should be noted that all property values are taken as the local values of the mixtures.

The energy equation can be obtained by considering the time rate of change of energy content, the convection of energy, and the flow of work. Neglecting differences between component kinetic energies gives

$$\vec{v} \cdot \left[ \rho \vec{V} \left( h + \frac{V^2}{2} \right) - \bar{\tau} \vec{V} + \vec{Q} \right] = 0 \quad (9)$$

where

$\bar{\tau}$  = viscous stress tensor

$\vec{Q}$  = energy flow vector

The second term in the brackets describes the diffusion of work due to the action of the stress field; the last term describes the diffusion of energy rather than heat. In a multicomponent system careful distinction should be made between heat flow  $\vec{q}$  and energy flow  $\vec{Q}$ , which includes the effect of the diffusing components.

Introducing the equations of motion and continuity results in the following simplification:

$$\rho \vec{v} \cdot \vec{v} - \vec{v} \cdot \nabla p + \nabla \cdot \vec{q} - \frac{1}{T} \cdot \nabla \vec{v} = 0 \quad (10)$$

Now, it may be shown in classical thermodynamics that the rate of change of enthalpy is equal to the change of heat occurring through any thermal transfer mechanism (closed system) plus the heat change brought about by concentration variations (open system):<sup>(5)</sup>

$$\frac{dh}{dt} = T \frac{ds}{dt} + \sum_i \mu_i \frac{dc_i}{dt} - \frac{1}{\rho} \frac{dp}{dt} \quad (11)$$

where

$\mu_i$  = chemical potential of  $i^{\text{th}}$  component

$S$  = entropy of the system

In order to solve Eq. (11), the first two terms on the right must be evaluated or expressed more directly in the properties of the system. In irreversible thermodynamics<sup>(6)</sup> it has been shown that an entropy continuity equation may be written:

$$\rho \frac{ds}{dt} + \nabla \cdot \vec{s} = \sigma \quad (12)$$

where

$\vec{s}$  = entropy flux vector



$\sigma$  = rate of irreversible entropy production

The term on the left indicates the total change of entropy within the system.  $V \cdot \bar{s}$  is the reversible flow of entropy through the bounding surface and  $\sigma$  is the rate of irreversible entropy production.

The reversible entropy flow vector is given by

$$\vec{T_s} = \vec{\dot{Q}} - \sum_1 \mu_1 \vec{j}_1 \quad (13)$$

and

$$\vec{\dot{Q}} = \vec{\dot{q}} + \sum_1 h_1 \vec{j}_1 \quad (14)$$

where

$h_1$  = partial enthalpy of 1<sup>th</sup> component

Combining Eqs. (10), (11), (12), and (13) we obtain the following expression for the rate of irreversible entropy production:

$$T\sigma = - \vec{\dot{q}} \cdot \nabla \ln T - \sum_1 \vec{j}_1 \cdot \vec{\bar{X}}_1 + \vec{\bar{s}} \cdot \nabla \vec{V} \quad (15)$$

where

$$\vec{\bar{X}}_1 = T \nabla \left( \frac{\mu_1}{T} \right) + \frac{h_1}{m_1} \nabla \ln T \quad (16)$$

The kinetic theory of dilute gases<sup>(7)</sup> gives expressions for the flux vectors for energy and diffusion:

$$\vec{\dot{q}} = - \lambda' \nabla T - p \sum_j \frac{D_j^T}{n_j m_j} \vec{\dot{d}}_j \quad (17)$$

$$\vec{j}_1 = - \frac{D_1^T}{T} \nabla T + \frac{n^2}{\rho} \sum_j m_1 m_j D_{1j} \vec{d}_j \quad (18)$$

The  $D_{1j}$  are generalized diffusion coefficients, the  $D_1^T$  are generalized thermal diffusion coefficients, and  $\lambda'$  is the "rest" thermal conductivity.\* The  $\vec{d}_1$  is given as

$$\vec{d}_1 = \frac{n_1}{p} \sum_j \left( \frac{\partial \mu_1}{\partial n_j} \right)_{T,p} \nabla n_j + (n_1 \bar{v}_1 - c_1) \nabla \ln p \quad (19)$$

where

$\bar{v}_1$  = the partial molal volume

For a mixture of ideal gases, Eq. (19) becomes

$$\vec{d}_1 = \frac{n_1}{p} \left( - \frac{kT}{n} \right) \sum_{j1} \nabla n_j + \left( \frac{n_1}{n} - c_1 \right) \nabla \ln p \quad (19a)$$

For a perfect gas, Eq. (19a) reduces to

$$\vec{d}_1 = R_1 T \nabla \ln p + R_1 T \nabla \ln \left( \frac{n_1}{n} \right) \quad (19b)$$

Substituting  $\vec{d}_1$  into Eq. (15) and noting  $\sum_1 \vec{d}_1 = 0$  results in  $\vec{d}_1 = \vec{X}_1$ . Equations (17) and (18) indicate a linear relationship between the fluxes  $\vec{q}$  and  $\vec{j}$  and the "forces"  $\nabla \ln T$  and  $\vec{X}_1$ . The Onsager reciprocity relations which are valid for such linear relations give

$$D_1^T = D_j^T \quad D_{1j} = D_{j1} \quad (20)$$

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\* This is the thermal conductivity which describes heat transfer solely by conduction without the convective effects of the induced thermodiffusion flow.

Finally, using Eq. (4) and eliminating  $\vec{d}_1$  from Eqs. (17) and (18) permits us to write

$$\frac{\vec{\lambda}}{q} = -kVT + RT \sum_{i \neq j} k^T \frac{\vec{j}_i}{c_i c_j} \quad (21)$$

and

$$\vec{j} = \rho \sum_{i \neq j} D_{ij} \left[ \frac{c_i c_j}{RT} (\vec{d}_j - \vec{d}_i) - k^T v \ln T \right] \quad (22)$$

where  $k$  is the usual thermal conductivity and  $k^T$  is the thermal diffusion coefficient. It should be noted that  $k$  includes the conducting effects of the induced diffusion stream and does not reduce to  $\lambda'$  even for  $\vec{j} = 0$ .

$$\frac{\vec{\lambda}}{q} = -\lambda' VT \approx -kVT \quad (22a)$$

It might be noted that  $k$  and  $\lambda'$  differ by a negligibly small amount for materials with high molecular weights. If we are concerned with light particles such as electrons, then this difference becomes significant.

Since the problem concerns a two-component system, the equations will be simplified for this special case. The components are related by the following conditions:

$$\begin{aligned} c_1 + c_2 &= 1 \\ \vec{j}_1 + \vec{j}_2 &= 0 \end{aligned} \quad (23)$$

Then the equations can be put in the form

$$\frac{\partial \rho}{\partial t} + v \cdot \rho \vec{v} = 0 \quad (2)$$

$$\rho \frac{dc}{dt} + v \cdot \vec{j} = 0 \quad (24)$$

$$\rho \frac{d\vec{V}}{dt} = -\nabla p + \nabla \cdot \left( k - \frac{2}{3} \mu \right) \vec{\nabla} \cdot \vec{V} + \mu (\nabla \vec{V} + \vec{V} \nabla) \quad (8)$$

$$\frac{\lambda}{q} = -kVT + \frac{M^2}{M_1 M_2} \alpha RT \bar{J}_1 \quad (25)$$

where

$$\bar{J}_1 = -\rho D_{12} \left[ v c_1 + \frac{M_2 - M_1}{M} c_1 (1 - c_1) v \ln p + \alpha c_1 (1 - c_1) v \ln T \right]$$

$$\alpha = \frac{k^T}{c_1 c_2}$$

### III. BOUNDARY-LAYER EQUATIONS

Consider the problem of a binary boundary layer on a flat plate. The equations governing the flow of a single component gas over a flat plate are well known and have been extensively investigated. A solution of the binary-boundary-layer equations will require a careful order-of-magnitude analysis to determine if any simplifications are permissible. This has been done by Hall.<sup>(3)</sup> The analysis of the mass-diffusion will be repeated here because of its importance in the problem.

Let two new variables  $X$  and  $Y$  be defined and nondimensionalize  $u$ ,  $v$ , and  $c$  as follows:

$$\bar{u} = \frac{u}{u_e}, \quad \bar{v} = \frac{v}{u_e}, \quad X = \frac{x}{L}, \quad Y = \frac{y}{\delta}, \quad \bar{c}_1 = \frac{c_1}{c_{1w}} \quad (26)$$

$L$  = characteristic dimension (x-direction)

$\delta$  = thickness of boundary layer

$u$  = velocity in the x-direction

$v$  = velocity in the y-direction

$c_{1w}$  = concentration at the wall

$e$  = properties referred to edge of boundary layer

Substitution of new variables into Eq. (24) results in the transformation

$$\begin{aligned} & u_e \bar{L} \left\{ \bar{\rho} \bar{u} \frac{\partial \bar{c}_1}{\partial X} \right\} + u_e \bar{\delta} \left\{ \bar{\rho} \bar{v} \frac{\partial \bar{c}_1}{\partial Y} \right\} \\ &= D_{12} \frac{1}{\delta^2} \frac{\partial}{\partial Y} \left\{ \bar{\rho} \left[ \frac{\partial \bar{c}_1}{\partial Y} + \frac{M_2 - M_1}{M \rho_e} \bar{c}_1 (1 - \bar{c}_1) c_{1w} \frac{\partial \ln \rho}{\partial Y} + \frac{\alpha c_{1w}}{\rho_e} \bar{c}_1 (1 - \bar{c}_1) \frac{\partial \ln T}{\partial Y} \right] \right\} \\ &+ D_{12} \frac{1}{L^2} \frac{\partial}{\partial X} \left\{ \bar{\rho} \left[ \frac{\partial \bar{c}_1}{\partial X} + \frac{M_2 - M_1}{M \rho_e} \bar{c}_1 (1 - \bar{c}_1) c_{1w} \frac{\partial \ln \rho}{\partial X} + \frac{\partial c_{1w}}{\rho_e} \bar{c}_1 (1 - \bar{c}_1) \frac{\partial \ln T}{\partial X} \right] \right\} \end{aligned} \quad (27)$$

The two terms on the left side of Eq. (22) will be compared first. An order-of-magnitude analysis of the continuity equation indicated that  $v = O(\frac{\delta}{L} u)$ . This implies that both terms are of  $O(\frac{u}{L})$ . Since it is postulated that  $L \gg \delta$ , only the first term on the right side of  $O(\frac{D_{12}}{\delta^2})$  will be important under the conditions stated. If a diffusion boundary layer is to exist, both sides must be of similar order of magnitude:

$$O(\frac{u}{L}) = O(\frac{D_{12}}{\delta^2}) \quad (28)$$

This may be rewritten as

$$O(\frac{Re}{L^2}) = O(\frac{1}{\delta^2 Sc}) \quad (29)$$

If the diffusion boundary layer is to be similar to the momentum boundary layer, then it is evident that

$$Sc = O(1) \quad (30)$$

Nothing further can be stated concerning the terms within the brackets until more information is available about the derivatives and their coefficients. Consequently, the final form of the diffusion-boundary-layer equation is

$$\rho u \frac{\partial c_1}{\partial x} + \rho v \frac{\partial c_1}{\partial y} = \frac{\partial}{\partial y} \left\{ \rho D_{12} \left[ \frac{\partial c_1}{\partial y} + \alpha c_1 (1 - c_1) \frac{\partial \ln T}{\partial y} \right] \right\} \quad (31)$$

The other boundary-layer equations are

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \quad (32)$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) \quad (33)$$

$$\frac{\partial p}{\partial y} = 0 \quad (34)$$

$$\rho c_p u \frac{\partial T}{\partial x} + \rho c_p v \frac{\partial T}{\partial y} - u \frac{\partial p}{\partial x} = \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + \mu \left( \frac{\partial u}{\partial y} \right)^2 \quad (35)$$

$$- \frac{\partial}{\partial y} \frac{M^2}{M_1 M_2} \alpha p_{12} J + \rho D_{12} (c_{p_2} - c_{p_1}) \frac{\partial T}{\partial y} J$$

$$J_1 = \frac{\partial c_1}{\partial y} + \alpha c_1 (1 - c_1) \frac{\partial}{\partial y} \ln T \quad (36)$$

If we assume that the similarity condition applies, i.e., that the dependence upon  $x$  and  $y$  can be combined into a single variable giving the normal position with a scale adjustment according to the position along the plate, then the system of partial differential equations may be transformed into a set of ordinary differential equations. The variable  $\eta$  is defined as a dimensionless quantity by

$$\eta = \frac{1}{2} \frac{y}{\sqrt{x}} \sqrt{\frac{\rho_e u_e}{\mu_e}} = \frac{1}{2} \frac{y}{x} \sqrt{Re_{e_x}} \quad (37)$$

It should be noted that the introduction of this parameter may restrain the functional dependence of the unknowns. For instance, it will be shown later that  $v(x)$  must be proportional to  $1/\sqrt{x}$  if the equations are to exhibit unidependency. For the case of sublimation or evaporation, the species-conservation equation at the surface yields the required  $v(x)$  proportionality. Furthermore we define a stream function

$$\psi = f \sqrt{\mu_e u_e x / \rho_e} \quad (38)$$

which satisfies the continuity equation

$$\rho_e \frac{\partial \psi}{\partial y} = \rho u, \quad \rho_e \frac{\partial \psi}{\partial x} = -\rho v \quad (39)$$

If these functions are substituted into the boundary-layer equations, they take the form

$$r \frac{d}{d\eta} \left( \frac{f'}{\rho} \right) + \frac{d}{d\eta} \left[ \frac{\mu}{\mu_e} \frac{d}{d\eta} \left( \frac{f'}{\rho} \right) \right] = 0 \quad (40)$$

$$r \frac{dc_1}{d\eta} = \frac{dj_1}{d\eta} \quad j_1 = - \frac{\rho D_{12}}{\mu_e} \left[ \frac{dc_1}{d\eta} + \alpha c_1 (1 - c_1) \frac{d}{d\eta} \ln T \right] \quad (41)$$

$$r c_p \frac{dT}{d\eta} + \frac{1}{4} u_e^2 \frac{\mu}{\mu_e} \left[ \frac{d}{d\eta} \left( \frac{\rho_e}{\rho} f' \right) \right]^2 + \frac{d}{d\eta} \left[ \frac{k}{\mu_e} \frac{dT}{d\eta} - \frac{M^2}{M_1 M_2} \alpha R T j \right] + (c_{p2} - c_{p1}) j \frac{dT}{d\eta} = 0 \quad (42)$$

Because the system of equations is of the seventh order, it will be necessary to provide seven boundary conditions. These will vary with the type of mass transfer that we are interested in. Eckert, Schneider, and Köhler have solved the problem for the case of injection of hydrogen into a stream of air.<sup>(1)</sup> At the outer limits of the boundary layer, the conditions will always be the same:

$$\begin{aligned} y &= e & T &= T_e \\ u &= u_e & & \\ c_1 &= 0 & & \end{aligned} \quad (43)$$

In terms of our new variables, this appears as

$$\begin{aligned} \eta &= e & f' &= 2 \\ T &= T_e & & \\ c_1 &= 0 & & \end{aligned} \quad (44)$$

These equations have been solved by Eckert for the case of a porous plate. He found that by injecting a small amount of lightweight, high-heat-capacity gas, it is possible to reduce considerably the amount of heat removal from



the plate. His equations were simplified in the sense that he assumed the thermal diffusion coefficient to be zero. He did not investigate the stability characteristics of the system.

The complexity of the system of equations which must be solved in order to investigate the velocity, temperature, and concentration profiles is such that it is necessary to perform a machine calculation. First, the equations must be put into a system of first-order Runge-Kutta equations. We let

$$\begin{aligned}
 f &= y_0 & T &= y_3 & c_1 &= y_5 \\
 \frac{f'}{\rho} &= y_1 & T' &= y_4 & c_1' &= y_6 \\
 \bar{w}_1' &= y_2 & T'' &= y_4' & c_1'' &= y_6' \\
 \left[ \bar{w}_1' \right]' &= y_2'
 \end{aligned} \tag{45}$$

The boundary-layer equations may be put in the following forms:

$$A_{1j}y_{2j} = D_1 \tag{46a}$$

where

$$A_{11} = 1 \tag{46b}$$

$$A_{12} = 0 \tag{46c}$$

$$A_{13} = 0 \tag{46d}$$

$$A_{21} = 0 \tag{46e}$$

$$A_{22} = \frac{\alpha y_5}{y_3} (1 - y_5) \tag{46f}$$

$$A_{32} = 1 \tag{46g}$$

$$A_{31} = 0 \quad (46h)$$

$$A_{32} = \left\{ \frac{K}{N_{Pr_e}} \cdot \frac{N_{sc}}{\rho} - \frac{M\bar{R}}{M_1 M_2} \frac{\alpha^2}{c_{p_e}} y_5 (1 - y_5) \right\} \quad (46i)$$

$$A_{33} = - \frac{M\bar{R}}{M_1 M_2} \frac{\alpha}{c_{p_e}} y_3 y_6' \quad (46j)$$

$$D_1 = - \frac{y_0 y_2}{\mu} \quad (46k)$$

$$D_2 = - y_0 y_6 \frac{N_{sc}}{\rho} + \left\{ y_6 + \alpha y_5 (1 - y_5) \frac{y_4}{y_3} \right\} \left\{ \frac{N_{sc}'}{N_{sc}} - 1 \right\} \\ - \left\{ \alpha y_6 (1 - y_5) \frac{y_4}{y_3} + \alpha' y_5 (1 - y_5) \frac{y_4}{y_3} - \alpha y_5 y_6 \frac{y_4}{y_3} \right. \\ \left. - \alpha y_5 (1 - y_5) \frac{y_4^2}{y_3^2} \right\} \quad (46l)$$

$$D_3 = - \frac{N_{sc}}{\rho} \bar{c}_p y_0 y_4 - \frac{1}{4} (\gamma_e - 1) M_e^2 \frac{N_{sc}}{\rho} \frac{y_2^2}{\mu} - \frac{K'}{N_{Pr}} y_4 \\ + \left\{ \frac{M\bar{R}}{M_1 M_2} \frac{\alpha}{c_{p_e}} y_4 + \frac{M\bar{R}}{M_1 M_2} \frac{\alpha'}{c_{p_e}} y_3 + \frac{M'\bar{R}}{M_1 M_2} \frac{\alpha}{c_{p_e}} y_3 \right\} \\ \left\{ - y_6 - \alpha y_5 (1 - y_5) \frac{y_4}{y_3} \right\} - \frac{M\bar{R}}{M_1 M_2} \frac{\alpha y_3}{c_{p_e}}$$

$$\left[ \left\{ \frac{N_{sc}'}{N_{sc}} - 1 \right\} \left\{ y_6 + \alpha y_5 (1 - y_5) \frac{y_4}{y_3} \right\} \right. \\ - \left\{ \alpha y_6 (1 - y_5) \frac{y_4}{y_3} + \alpha' y_5 (1 - y_5) \frac{y_4}{y_3} - \alpha y_5 y_6 \frac{y_4}{y_3} \right. \\ \left. \left. - \alpha y_5 (1 - y_5) \frac{y_4^2}{y_3^2} \right\} \right] + \bar{c}_{p12} y_4 \left\{ y_6 + \alpha y_5 (1 - y_5) \frac{y_4}{y_3} \right\} \quad (46m)$$

The system of equations contained in Eqs. (46a - m) appears in Runge-Kutta method as

$$\begin{aligned}
 y_0' &= y_1 \bar{\rho} & y_4' &= f_2(y_0 \dots y_6) \\
 y_1' &= y_2/\bar{\mu} & y_5' &= y_6 \\
 y_2' &= f_1(y_0 \dots y_6) & y_6' &= f_3(y_0 \dots y_6) \\
 y_3' &= y_4
 \end{aligned} \tag{47}$$

where  $f_1$ ,  $f_2$ , and  $f_3$  are obtained by solving the system of Eqs. (46a - m). Before solving, however, it is necessary to specify four other boundary conditions. At the surface there are three obvious conditions:

$$\begin{aligned}
 y &= 0 & u &= 0 \\
 & & T &= T_w \\
 & & c_1 &= c_{1w}
 \end{aligned} \tag{48}$$

To account for the other boundary condition, we observe that the convective velocity of the foreign gas passing through the temuous channels of the plate will be large compared with the mass velocity of the air. In other words, we say that the mass of air which enters the plate is negligible. Mathematically, the diffusion current for the air is given by

$$\vec{j}_2 = c_2 \rho \vec{v}_2 \tag{49}$$

According to the assumption that the mass velocity of the air must be zero

$$\vec{v} + \vec{v}_2 = 0 \quad \text{for } y \leq 0 \tag{50}$$

The surface diffusion current for the foreign gas becomes

$$j_w = (1 - c_{1w}) \rho_w v_w = - \rho_w D_{12} \left( \frac{\partial c_1}{\partial y} \right)_w \tag{51}$$

Transforming into our new variables gives

$$f_w = \frac{\bar{\rho}_w}{Sc(1 - c_{1w})} \left( \frac{dc_1}{d\eta} \right)_w \quad (52)$$

It should be noted that the situation is quite different for a subliming solid. In this case there is no question that the mass flow of air back through the plate is zero. It is important, however, that the flow of foreign gas is coupled to the temperature boundary layer by the heat of sublimation. This can be related most simply through the Clapeyron-Clausius equation

$$\frac{dp}{dT} = \frac{\Delta H_t}{(\Delta V)T} \quad (53)$$

If we are considering sublimation,  $p$  is the vapor pressure over the solid;  $\Delta V = V_g - V_s \approx V_g$ , since  $V_g > V_s$ ; and  $\Delta H_t$  is the molar heat of sublimation. Now if we assume that the gas is ideal and that the heat of sublimation is a function of temperature only, we obtain

$$\ln p_w = \frac{1}{R} \int_{T_1}^{T_w} \frac{\Delta H_t}{T^2} dT \quad (54)$$

where  $T_1$  is the temperature at which the pressure is 1 atm.

The introduction of a subliming solid complicates the boundary conditions, because the wall temperature and concentration are no longer variable. For the case of transpiration cooling under a given set of external flow conditions, the mass injection rate (a variable boundary input) controls the concentration of the foreign gas at the surface and this in turn permits the choice of a variable surface temperature or heat transfer. Sublimation cooling imposes another condition, namely, a relationship between the surface temperature and concentration. The heat of sublimation fixes

this dependence as shown in Eq. (54). Furthermore, the heat transfer at the wall is related to the heat of sublimation in the following way (assuming no radiation or conduction losses):

$$(\rho v)_w = \frac{q_w}{\Delta H_s} \quad (54a)$$

It should be noted that the  $q_w$  takes into account the last blocking action of the foreign gas diffusing through it. The injection rate, therefore, is not arbitrary as in the previous case but is a function of heat flux and heat of sublimation. Choosing a material essentially fixes the heat of sublimation (except for a small dependence on temperature). A wall temperature now determines the wall concentration, hence the injection rate and finally the heat transfer. This heat-transfer rate must satisfy Eq. (54a).

The problem of handling the complex boundary conditions can be most easily taken care of by setting up another set of linear equations which must be solved simultaneously with the original system. The Runge-Kutta method requires only initial conditions, and so a series of initial guesses as to the original conditions will be required. The guesses will lead to a set of solutions which may then be compared with the boundary conditions at the outer edge of the boundary layer. A perturbation of the initial guesses and an examination of the results of the perturbation on the end conditions will then permit a linear compensator to adjust the initial guesses until the final boundary conditions are met.

#### IV. BOUNDARY-LAYER STABILITY

The solution to the boundary-layer equations having been obtained, it will be necessary to investigate the stability characteristics of the laminar compressible boundary layer with sublimation. Lin and Dunn have shown that an order-of-magnitude analysis of the complete linearized equations for a three-dimensional disturbance imposed upon a two-dimensional boundary layer indicates that they can be reduced to much simpler terms in the first approximation.<sup>(8)</sup> These simplified equations are still valid at high free-stream Mach numbers. However, the accuracy of a first approximation may not be great, so that higher-order approximations may have to be considered. Lin and Dunn also found that the conclusion reached by Lin and Lees, namely, that the relationship for the characteristic values is independent of the temperature fluctuations and the boundary conditions imposed upon them, is restricted in general validity to subsonic and slightly supersonic Mach numbers. At high Mach numbers the relation for the characteristic values depends in general upon the thermal boundary conditions. Furthermore, Lin and Dunn have developed a method for determining the stability characteristics of a compressible boundary layer which they feel is quite accurate up to a Mach number of two. It is believed that their results can be extended to Mach-number values as high as six.

The work of Lin and Dunn was not concerned with binary compressible boundary layers. Theoretically, a new set of equations must be obtained including the diffusion equation. The mathematical difficulties of solving such a system of equations, even though they are linearized, is very great. Instead, as a first approximation, it might be possible to use the velocity

and temperature profiles obtained from the solution of the above equations in the Lin and Dunn method. This involves assuming that the concentration at the surface is exceedingly small, that the diffusion velocity is negligible, and that the effect of the foreign material in the boundary layer, as far as stabilization is concerned, will be evident only on the basis of the velocity and temperature profiles. This assumption has been shown to be correct by E. E. Covert.<sup>(9)</sup>

# V. NUMERICAL SOLUTION

Before Eqs. (46a - m) can be solved, it will be necessary to indicate some relationship between the thermal properties and the temperature and concentration at any given point. The boundary conditions must be more carefully defined. Equation (52) accurately defines the relationship at the wall between the velocity (of the foreign substance) issuing from the wall and the concentration of the foreign substance at the wall. However, in the case of a binary boundary layer, the temperature difference existing at any point also causes a diffusion of material. Of course, this is also true at the surface, and therefore the boundary condition of Eq. (52) is no longer quite correct. Instead, it can be shown that the influence of the temperature gradient on the mass transfer at the wall can be taken into account by including a thermal-diffusion term in our original equation:

$$f_w = \frac{\bar{\rho}_w}{N_{sc}(1 - c_{1w})} \left( \frac{dc_1}{d\eta} \right)_w + \alpha_{c1w} (1 - c_{1w}) \frac{1}{T_w} \left( \frac{dT}{d\eta} \right)_w \quad (55)$$

Equation (54) can be rearranged so that it is more amenable to calculation. If the total pressure is  $p_0$  and the heat of sublimation is a function (quadratic) of temperature only, the equation can be put in the form

$$\ln N_1 = R \int_{T_1}^{T_w} \frac{1}{T^2} (S_1 + S_2 T + S_3 T^2) dT + k_1 \quad (56)$$

If it is assumed that the wall concentration will always remain small, then Eq. (56) can be integrated and rearranged to give



$$\ln c_{1v} = \frac{1}{R} \left[ s_1 \left( \frac{T_v - T_1}{T_v T_1} \right) t_e + s_2 \ln \frac{T_v}{T_1} + s_3 t_e (T_v - T_1) \right] + k_2$$

$$k_2 = - \ln \frac{M_2 p_e}{M_1} \quad (57)$$

The seven boundary conditions may now be brought together and put into the new coordinate system as follows:

$$y_1(\infty) = 2 \quad (58a)$$

$$y_5(\infty) = 0 \quad (58b)$$

$$y_3(\infty) = 1 \quad (58c)$$

$$y_1(0) = 0 \quad (58d)$$

$$y_5(0) = c_{1v} \quad (58e)$$

$$y_o(0) = \frac{\bar{p}(0)}{N_{sc}(0)1 - y_5(0)} \left[ y_5(0) + \alpha(0)y_5(0) \left\{ 1 - y_5(0) \right\} \frac{y_4(0)}{y_3(0)} \right] \quad (58f)$$

$$\ln y_5(0) = \frac{1}{R} \left[ s_1 \left\{ \frac{y_3(0) - y_3(1)}{y_3(0)y_3(1)} \right\} t_e + s_2 \ln \frac{y_3(0)}{y_3(1)} \right. \\ \left. + s_3 \left\{ y_3(0) - y_3(1) \right\} \right] - \ln \frac{p_e M_2}{M_1} \quad (58g)$$

The thermal properties must now be calculated in terms of the temperature and concentration at any given point. For reasons indicated elsewhere in this Memorandum, the diffusing substances chosen were iodine and carbon dioxide. The thermal properties of these materials were investigated, and those properties which were not available from experimental data were calculated using statistical mechanical formulas. In all cases, the properties were represented in a power form rather than the original formula, e.g., the Sutherland viscosity formula. In this way, the number of constants required was reduced to a minimum. All thermal properties have been normalized with respect to

air at the free-stream temperature. These normalized properties are represented in the following manner:

Density:

$$\bar{\rho} = \frac{1}{(1 + k_1 y_5) y_3} \quad (59)$$

Viscosity:

$$\bar{\mu} = \frac{k_2 y_3^{k_3} y_5}{(k_4 + k_5 y_5)} + \frac{k_6 y_3^{k_7} (1 - y_5)}{(k_8 + k_9 y_5)} \quad (60)$$

Schmidt number:

$$N_{sc} = \frac{k_{10}}{y_3^{k_{11}}} \quad (61)$$

Thermal diffusivity:

$$\alpha = k_{12} y_3 + k_{14} \quad (62)$$

Pure component thermal capacity:

$$\bar{c}_{p12} = k_{15} + k_{16} y_3 + k_{17} y_3^2 \quad (63)$$

Mixture thermal capacity:

$$\bar{c}_p = k_{18} + k_{19} y_5 + k_{20} y_3 + k_{21} y_3^2 + k_{22} y_3 y_5 + k_{23} y_5 y_3^2 \quad (64)$$

Mixture thermal conductivity:

$$\bar{k} = \frac{w_2 k_1 + w_1 k_2 + k_{24} + k_1 k_2}{w_1 w_2 + k_{25}}$$

$$k_1 = k_{26} y_3^{k_{27}}$$

$$k_2 = k_{28} y_3^{k_{29}}$$

$$w_1 = 1 + k_{30} \frac{1 - y_5}{y_5}$$

$$w_2 = 1 + k_{31} \frac{y_5}{1 - y_5} \quad (65)$$

Mixture molecular weight:

$$k \bar{M} = \frac{M_1 M_2}{M_1 + (M_2 - M_1) y_5} \quad (66)$$

Some of the values for these constants for the case of iodine vapor - air and carbon dioxide - air mixtures are given in Table 1. To solve the equations, the following constants must be specified:  $K_{1j}$ ,  $M_1$ ,  $R_1$ ,  $\bar{c}_{pe}$ ,  $t_e$ ,  $M_e$ , and  $N_{PR}$ . In addition, either the concentration or the temperature at the wall must be specified. The gradients of the temperature and concentration at the wall are also required, but unknown. It will be necessary to guess these in accordance with our iteration system so that the defined boundary conditions at the outer edge of the boundary layer result upon integration.

Table 1

CONSTANTS FOR MULTICOMPONENT PROPERTY EQUATIONS

| Constant                     | CO <sub>2</sub> -air |                    | Iodine-air          |                     |
|------------------------------|----------------------|--------------------|---------------------|---------------------|
| k <sub>1</sub>               | -0.341               |                    | -0.8858             |                     |
| k <sub>2</sub> <sup>a</sup>  | 0.855                |                    | 0.615 <sup>b</sup>  | 0.884 <sup>c</sup>  |
| k <sub>3</sub> <sup>a</sup>  | 0.764                |                    | 1.15 <sup>b</sup>   | 0.883 <sup>c</sup>  |
| k <sub>4</sub> <sup>a</sup>  | 1.2405               |                    | 2.3458              |                     |
| k <sub>5</sub> <sup>a</sup>  | -0.2405              |                    | -1.3458             |                     |
| k <sub>6</sub> <sup>a</sup>  | 1.125                |                    | 1.125               |                     |
| k <sub>7</sub> <sup>a</sup>  | 0.625                |                    | 0.625               |                     |
| k <sub>8</sub> <sup>a</sup>  | 1.2405               |                    | 1.0000              |                     |
| k <sub>9</sub> <sup>a</sup>  | -0.2405              |                    | -0.67053            |                     |
| k <sub>10</sub>              | 1.022 <sup>b</sup>   | 1.004 <sup>c</sup> | 1.933 <sup>b</sup>  | 1.613 <sup>c</sup>  |
| k <sub>11</sub>              | 1.78 <sup>b</sup>    | 1.66 <sup>c</sup>  | 1.87 <sup>b</sup>   | 1.7 <sup>c</sup>    |
| k <sub>12</sub>              | 0.63 <sup>b</sup>    | 0.115 <sup>c</sup> | 2.44 <sup>b</sup>   | 0.21 <sup>c</sup>   |
| k <sub>14</sub>              | -1.18 <sup>b</sup>   | 0.00 <sup>c</sup>  | -2.428 <sup>b</sup> | -0.768 <sup>c</sup> |
| k <sub>15</sub>              | -0.21                |                    | -0.755              |                     |
| k <sub>16</sub> <sup>a</sup> | 0.071                |                    | -0.071              |                     |
| k <sub>17</sub> <sup>a</sup> | -0.0046              |                    | 0.0038              |                     |
| k <sub>18</sub>              | 0.877                |                    | 0.877               |                     |
| k <sub>19</sub>              | -0.207               |                    | -0.755              |                     |
| k <sub>20</sub>              | 0.121                |                    | 0.125               |                     |
| k <sub>21</sub>              | -0.0036              |                    | -0.0048             |                     |
| k <sub>22</sub>              | -0.075               |                    | -0.075              |                     |
| k <sub>23</sub>              | -0.0048              |                    | 0.0048              |                     |
| k <sub>24</sub>              | 1.6202               |                    | 0.5271              |                     |
| k <sub>25</sub>              | -0.6563              |                    | -0.0695             |                     |

| Constant        | CO <sub>2</sub> -air     | Iodine-air  |
|-----------------|--------------------------|---|
| k <sub>26</sub> | 1.15 x 10 <sup>-2</sup>  | 1.15 x 10 <sup>-2</sup>   |
| k <sub>27</sub> | 0.859                    | 0.859   |
| k <sub>28</sub> | 6.074 x 10 <sup>-3</sup> | 1.09 x 10 <sup>-2</sup> <sup>b</sup> 1.56 x 10 <sup>-2</sup> <sup>c</sup> |
| k <sub>29</sub> | 1.396                    | 1.15 <sup>b</sup> 0.833 <sup>c</sup>                                      |
| k <sub>30</sub> | 2.0976                   | 7.676   |
| k <sub>31</sub> | 1.5361                   | 0.461   |

<sup>a</sup>Values obtained from experimental data.

<sup>b</sup>Dimensionless absolute temperature less than 3.

<sup>c</sup>Dimensionless absolute temperature more than 3.

## VI. STABILITY CHARACTERISTICS

As indicated previously, the stability characteristics of the diffusion systems will be examined by placing the temperature and velocity profiles obtained from the solution of Eq. (47) into the Lin and Dunn method.

The theory proposed by Lin and Dunn will not be presented here, but rather the numerical procedure to be used in calculating the neutral stability curve. First, we define two functions  $v(c)$  and  $\lambda(c)$  as follows:

$$v(c) = \frac{y_2(0)y_1(c)\pi}{\bar{\mu}(0)y_3(0)} \left\{ \frac{\bar{\mu}}{y_2^2} y_0 y_3 + \frac{y_4 \bar{\mu}^2}{y_2^2} \right\}_{u=c} \quad (67)$$

$$\lambda(c) = \frac{y_2(0)}{\bar{\mu}(0)} \frac{\bar{\mu}(0)^{1/2}}{[y_1(c)]^{3/2} \bar{\rho}(0)^{1/2}} \frac{3}{2} \int_0^{\eta_c} \sqrt{\frac{y_1(c) - y_1}{2\bar{v}}} d\eta - 1 \quad (68)$$

Given a  $y_3(0)$ ,  $v(c)$  and  $\lambda(c)$  can be determined for any  $y_1(c)$ . The following approximation scheme is suggested:

$$\phi_1(z^{n+1}) = \frac{(1 + \lambda)v}{(1 + \lambda u^n)^2 + \lambda^2 v^2} \quad (69)$$

$$u^{n+1} = \phi_r(z^{n+1}) \left\{ \frac{(1 + \lambda u^n)^2 + \lambda^2 v^2}{(1 + \lambda)(1 + \lambda u^n)} \right\} - \frac{\lambda v^2}{1 + \lambda u^n} \quad (70)$$

In Eqs. (69) and (70),  $\phi_1(z)$  and  $\phi_r(z)$  are known functions of  $z$ ; a tabulated set of these functions may be found in Lin and Dunn's report.<sup>(8)</sup> Since  $\lambda$  is usually small, initial guesses for the approximations scheme may be

$$\phi_1(z) = v(c) \quad u = \phi_r(z) \quad (71)$$

For a given value of  $y_3(0)$ ,  $v(c)$  and  $\lambda(c)$  may be determined for a given  $y_1(c)$ . Now  $v(c)$  defines an initial  $z$  which is used to obtain an initial value of  $u$ . From Eq. (69),  $\phi_1(z^{n+1})$  is calculated. The  $z^{n+1}$  and  $u^n$  are put into Eq. (70) to calculate a new  $u^{n+1}$ . In this way, it is possible to obtain a solution  $(z,u)$  for each value of  $y_1(c)$ . The wave number is now calculated from

$$\frac{k}{\alpha} = k_{H_1}^2 + \frac{4y_3(0) \mu(0)}{8y_1(c)y_2(0)} \left\{ u(e) + \frac{B_e^2 y_1(c)y_2(0)}{4y_3(0) \mu(0)} - \left[ 1 + \frac{y_1(c)}{2} (k_{10} + k_{11} + k_{12}) \right] \right\} \quad (72a)$$

where

$$k = \frac{\sqrt{1 - M_e^2 [1 - y_1(c)/2]^2}}{[1 - y_1(c)/2]^2} \quad (72b)$$

and

$$B = \frac{\delta}{2x} \sqrt{Re_x} \quad (72c)$$

$H_1$ ,  $k_{10}$ ,  $k_{11}$ , and  $k_{12}$  may be computed by machine as indicated in Appendix B. The Reynolds number may now be obtained from

$$\alpha Re_\delta = \frac{z^3}{9I^2} B^2 [y_2(0)]^2 \frac{y_3(0)}{\mu(0)} \quad (73a)$$

where

$$I(c) = \int_0^{\frac{y_1(c)}{2}} \sqrt{\frac{[y_1(c) - y_1] y_3(0) \mu}{8 y_3 \mu(0)}} \frac{dy_1}{s} \quad (73b)$$

## VII. SIMPLIFIED SOLUTION

Even a cursory examination of the equations to be solved would indicate the difficulty involved in making reasonable approximations to the initial values of the system which the Runge-Kutta routine requires. The stability of the Convergence Scheme was not studied; but it may be presumed that if the initial values are not at least "approximately" close to the proper values, then it may be that the routine is nonconverging and will never give a solution. To prevent such difficulties, it might be judicious to solve a simplified case first so that the initial conditions resulting from this attempt may be used in the more complex solution. Time limitations also played an important part in choosing the simplified case.

If we presume that the fluid is incompressible and that no heat transfer is present, the equations presented previously reduce to

$$f \frac{d}{d\eta} \left( \frac{1}{\rho} \frac{df}{d\eta} \right) + \frac{d}{d\eta} \left[ \bar{\mu} \frac{d}{d\eta} \left( \frac{1}{\rho} \frac{df}{d\eta} \right) \right] = 0 \quad (74)$$

$$\frac{d}{d\eta} \left[ \frac{1}{N_{sc}} \bar{\rho} \frac{dc_1}{d\eta} \right] + f \frac{dc_1}{d\eta} = 0 \quad (75)$$

The boundary conditions for this fifth-order, ordinary system of differential equations are

$$\left. \begin{array}{l} f' = 0 \end{array} \right\} \quad (76a)$$

$$\left. \begin{array}{l} c_1 = c_{1w} \end{array} \right\} = 0 \quad (\text{wall concentration}) \quad (76b)$$

$$\left. \begin{array}{l} f = f_w \end{array} \right\} \quad (\text{injection parameter}) \quad (76c)$$

$$\left. \begin{array}{l} f' = 2 \end{array} \right\} \quad (76d)$$

$$\left. \begin{array}{l} c_1 = 0 \end{array} \right\} = \infty \quad (76e)$$



To use the Runge-Kutta system, all boundary conditions will have to be specified at the wall, i.e.,  $y_1(0)$ . Two initial conditions are definitely known, the velocity at the wall is zero, and the concentration here may be specified. It will be necessary to fulfill end boundary conditions such that concentration and the concentration gradient become zero at the outer edge of the boundary layer. If, using Eqs. (74) and (75), we define

$$y_0 = f \quad (77a)$$

$$y_1 = 1/\rho \, df/d\eta \quad (77b)$$

$$y_2 = \bar{\mu} \, d/d\eta \left[ 1/\rho \, df/d\eta \right] \quad (77c)$$

$$y_3 = c_1 \quad (77d)$$

$$y_4 = \bar{\rho} \, dC_1/d\eta \quad (77e)$$

then the system of equations appears in the desired linear form

$$y'_0 = \bar{\rho} \, y_1 \quad (78a)$$

$$y'_1 = y_2 \sqrt{\mu} \quad (78b)$$

$$y'_2 = -y_0 y_2 \sqrt{\mu} \quad (78c)$$

$$y'_3 = y_4 \sqrt{\rho} \quad (78d)$$

$$y'_4 = -N_{sc} y_0 y_4 \sqrt{\rho} \quad (78e)$$

with the boundary conditions

$$y_1(0) = 0 \quad (79a)$$

$$y_3(0) = c_{1w} \quad (79b)$$

$$y_0(0) = f_w \quad (79c)$$

$$y_1(e) = 2 \quad (79d)$$

$$y_3(e) = 0 \quad (79e)$$

Knowing  $y_1(0)$  and  $y_3(0)$ , we find it necessary to choose  $y_0(0)$  and  $y_2(0)$  in such a way that the last two boundary conditions are fulfilled. A convergence scheme was used which depended upon a grid of guesses which was then

interpolated to give the proper initial values. The solution of these equations (exactly similar to those of Eckart<sup>(10)</sup>) was carried out for the cases of iodine and carbon dioxide diffusing through the boundary layer. The equations yielded velocity and concentration profiles, as well as relationships between the drag coefficient and wall concentration or inlet velocity. Finally, for each of the wall concentrations specified, accurate initial conditions were obtained which would undoubtedly serve as a good jumping-off place for a convergence scheme if the complex equations are solved.

Once having obtained the velocity and concentration profiles, we must not forget that the original purpose of the investigation was to examine the stability character of diffusion boundary layers. Here again, a simplified approach will be used. It was shown in the Tollmien-Rayleigh analysis that the curvature of the velocity profile was fundamental in indicating the stability of an incompressible flow. If the flow were in the nature of a Blasius profile, i.e., if it did not exhibit a point of inflection, the profile would be stable regardless of Reynolds number. If a profile showed a definite point of inflection, as in the case of flow against an adverse pressure gradient, the boundary layer would be unstable for any Reynolds number. Although this analysis holds only for the case of inviscid flow, it nevertheless indicates the importance of the velocity profile in determining, qualitatively at least, the stability characteristics of the flow. Van Driest has extended the analysis to the case of compressible fluids.<sup>(2)</sup> He showed that if the quantity

$$\left( \frac{d}{dy} \right) \left[ \rho \left( \frac{du}{dy} \right) \right] = 0 \quad c_u/u_\infty > 1 - 1/M_\infty \quad (80)$$

a neutral or self-excited disturbance appears in the boundary layer at that point. In general, the distribution of the quantity  $\bar{\rho} \frac{d}{d} \left[ \frac{u}{u_{\infty}} \right]$  is decisive in determining the stability of the boundary layer. This may be seen by examining  $\rho \frac{du}{dy}$  profiles for the cases of heating and cooling the boundary layer.<sup>(2)</sup> The cooling curves show a tendency toward leveling out, whereas the heating curves all exhibit the characteristic hump, i.e., the point of inflection. This, then, will be one way to examine the stability character of binary boundary layers.

Another way to characterize stability of flow is to indicate the minimum critical Reynolds number of a system. This is the smallest Reynolds number at which neutral disturbances can exist. It is obvious that we would like to make the minimum critical Reynolds number as large as possible. One way to obtain this Reynolds number is to observe a plot of  $\alpha$  (disturbance wavelength) against Reynolds number. The curve is usually paraboloid in shape and contains a region in which all self-excited disturbances are amplified. Outside this curve all disturbances are damped. The tip of the paraboloid will be the minimum critical Reynolds number. The calculation of such a curve is very tedious, and Lin<sup>(11)</sup> has suggested an approximation formula which has been shown to be within 20 per cent of the values obtained with the more accurate analyses. The formula is

$$R_{cr} = \frac{25 \left( \frac{du}{dy} \right)_w}{\left[ u(y_1) \right]^4} \quad (81)$$

where

$$\pi \left( \frac{du}{dy} \right)_w \left[ \frac{2 \left( \frac{du}{dy} \right)_w \cdot y_1}{u(y_1)} - 3 \right] \frac{u(y_1) u''(y_1)}{\left[ u'(y_1) \right]^3} = 0.58 \quad (82)$$

Lin's formula will facilitate a qualitative comparison of the stability of the three systems.

### VIII. RESULTS

The three systems investigated were distinguished by the foreign substance used in each: hydrogen, iodine vapor, or carbon dioxide. Hydrogen was chosen because it has been used in binary-boundary-layer analyses previously, mostly on the basis of its ability as a coolant. In this way it would act as a stabilizing factor. However, its light weight would cause it to act as a buoying agent, tending to "lift" the boundary layer from the surface. Needless to say, this effect would be destabilizing since it increases the boundary-layer thickness. The coolant effect of the hydrogen will not show up in the analysis, and therefore it might be expected that its stability character will not be as good as that of the heavier materials--carbon dioxide and iodine. These two were chosen on the basis of the Van Driest criteria. It was hoped that by making the quantity  $\rho du/dy$  as large as possible near the surface, the unstable hump might possibly be avoided. Also, the introduction of heavy molecules might be thought of as "cooling" the boundary layer simply by increasing the density at the region near the wall.

The equations were solved as indicated, and the numerical results are presented in the Appendices. An examination of the velocity and concentration profiles bears out the theory that hydrogen acts as a buoying agent while carbon dioxide and iodine act in the opposite direction. The hydrogen diffusing rapidly through the boundary layer slows up the x-directed velocity of the airstream. In Fig. 1, for a wall concentration of hydrogen of 0.2, only 95 per cent of the free-stream velocity has been reached at  $\eta = 3$ . For the same conditions, the free-stream velocity for carbon dioxide and iodine injection is reached at  $\eta = 2.7$  and at  $\eta = 2.8$ , respectively

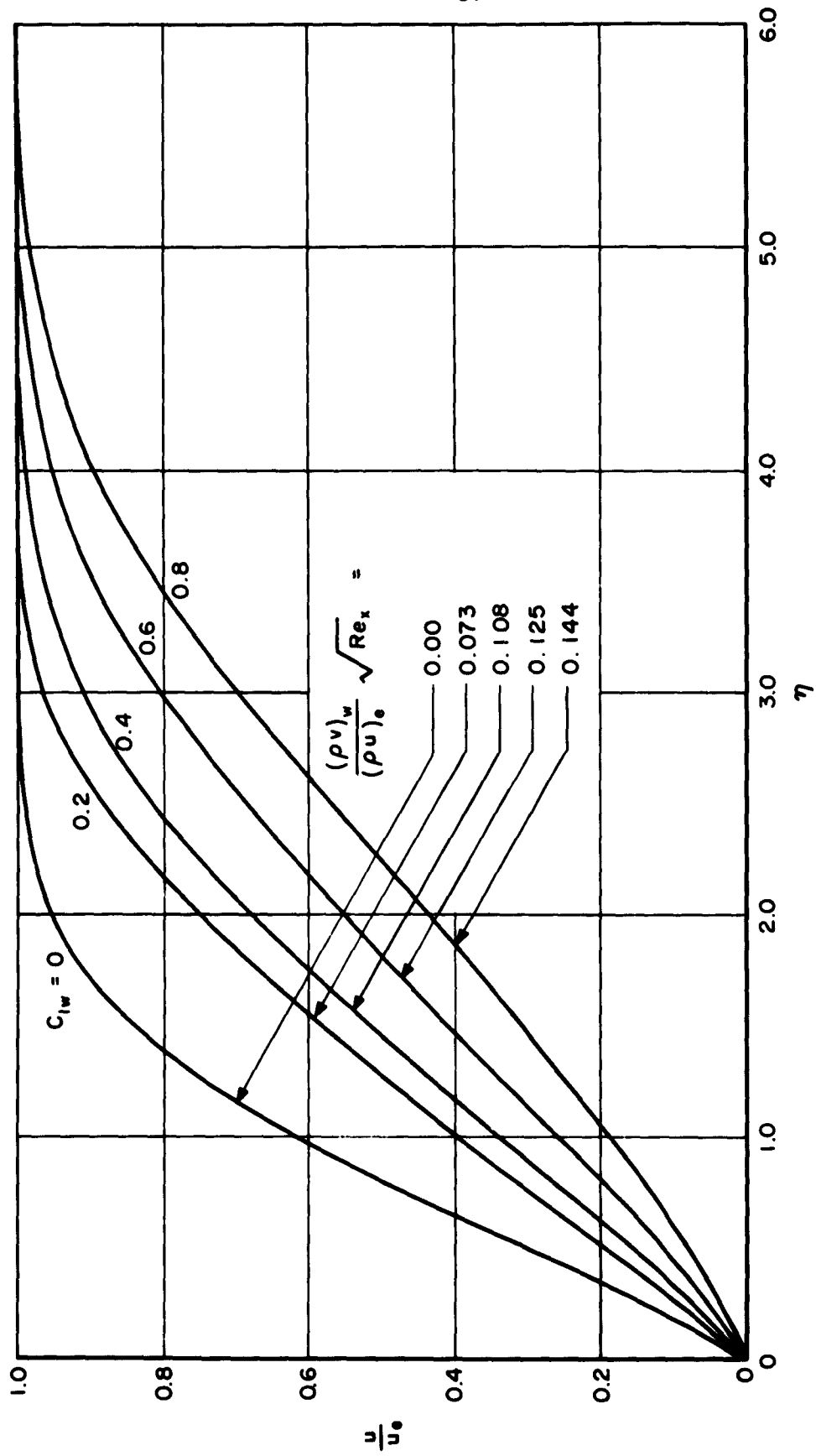


Fig. 1 — Velocity profiles: hydrogen

(see Figs. 2 and 3). The same characteristic of hydrogen is borne out by the concentration profiles. Hydrogen concentrations extend out into the boundary layer well past  $\eta = 3$  (Fig. 4), whereas the concentration of carbon dioxide and iodine for  $\eta = 3$  are practically zero (Figs. 5 and 6).

Equation (55) gives the relation between the injection rate and the concentration of the foreign material. Because of its high diffusion coefficient and low density, hydrogen should have a high velocity and a low injection rate. Examination of Fig. 7 indicates that this is generally true. The injection rates of the three materials are almost the same at a wall concentration of 0.2; but it must be remembered that even at this point the presence of the foreign constituents is beginning to taper off, and for smaller concentrations we may expect the injection rate to drop to zero quickly. At the higher concentrations, however, it is seen that large differences occur in the injection rates of the three materials. The essential nature of the curves of Fig. 7 is similar to a plot of the density ratios of the three materials (Fig. 8). Therefore, considering that the concentration gradients at the surface are almost the same for all materials, it is seen that the injection rate is dependent mainly upon the density ratio. At low concentrations all three density ratios approach unity, whereupon the Schmidt number or diffusion coefficient becomes the important factor. At extremely low wall concentrations the injection rates at the wall become almost proportional to the diffusion coefficients.

The curves in Fig. 9 showing the relationship between the drag coefficient and the injection rate (where  $f_w$  is a measure of injection rate, as shown by Eq. (52)) indicate that hydrogen is superior to carbon dioxide or iodine as a friction-reducing agent. If separation occurs when the

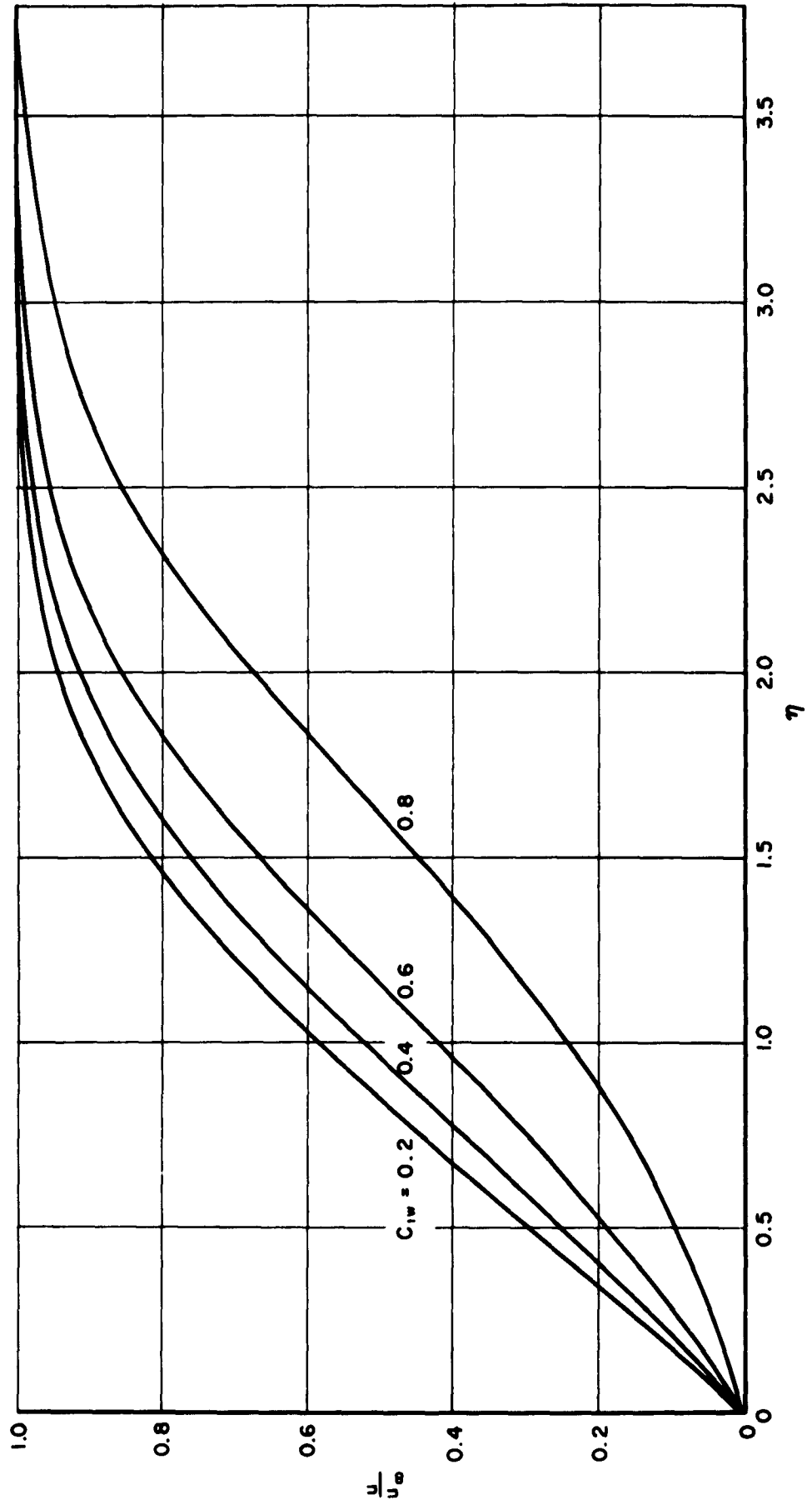


Fig. 2 — Velocity profiles: carbon dioxide



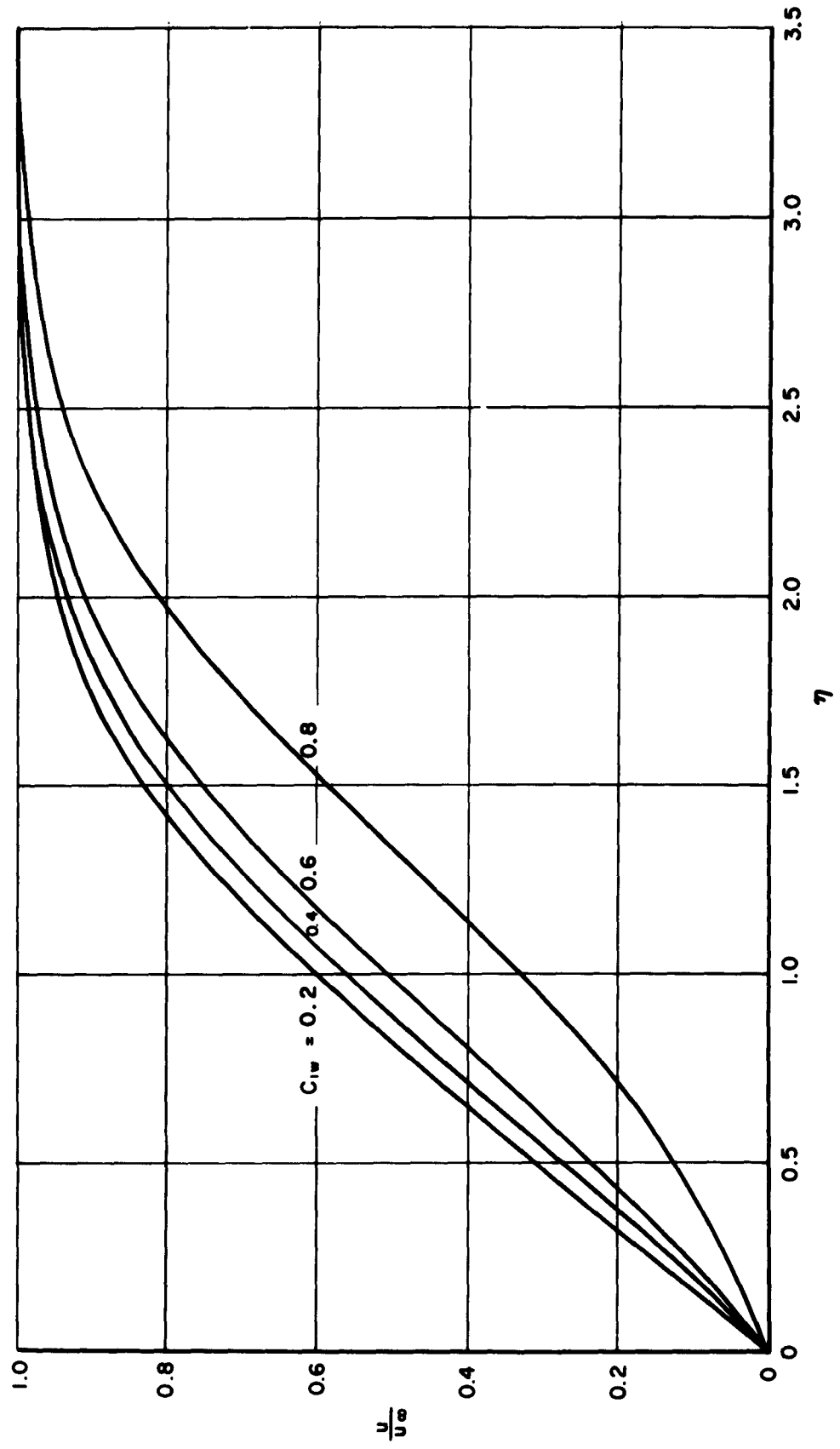


Fig. 3— Velocity profiles: iodine

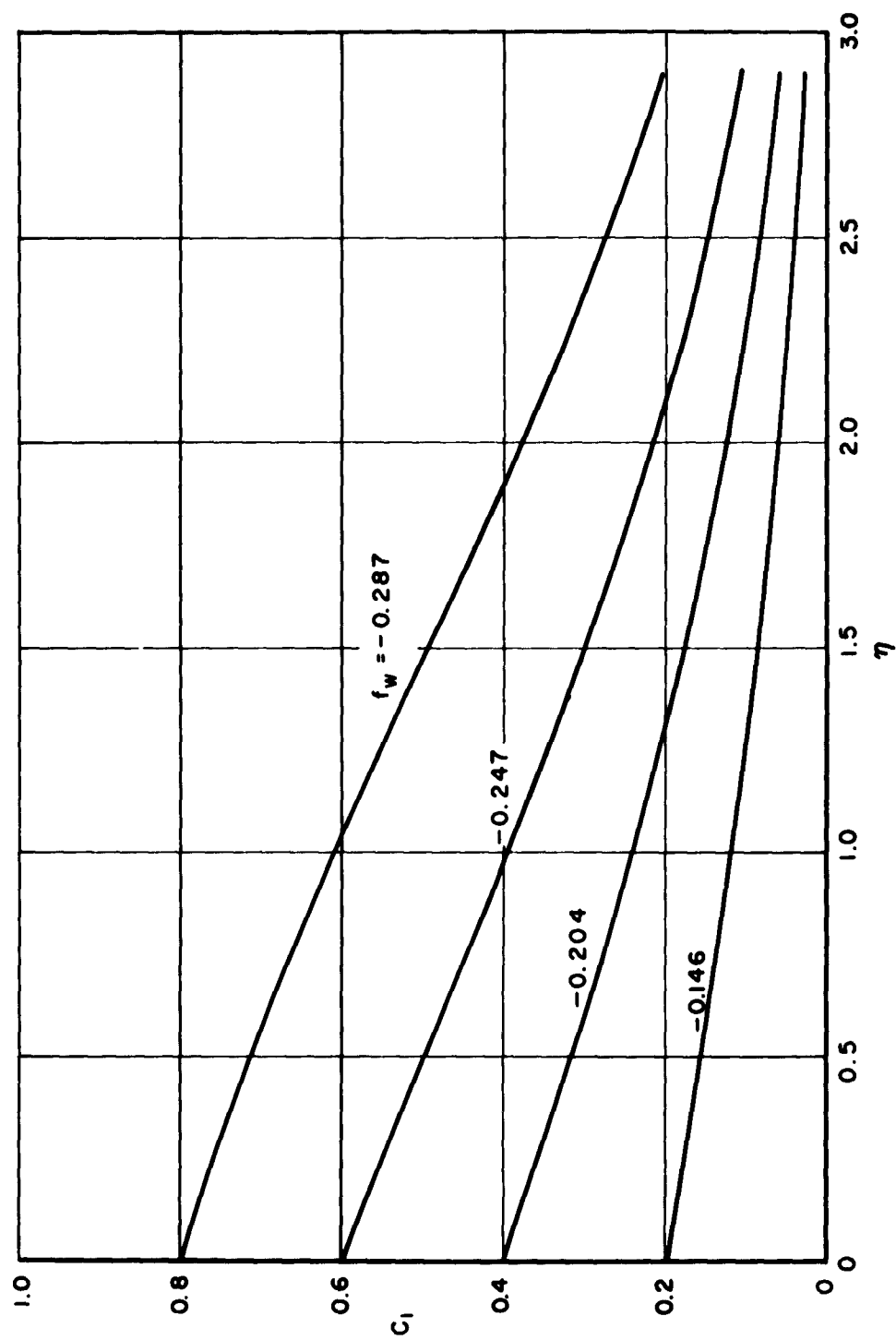


Fig. 4 — Concentration profiles: hydrogen

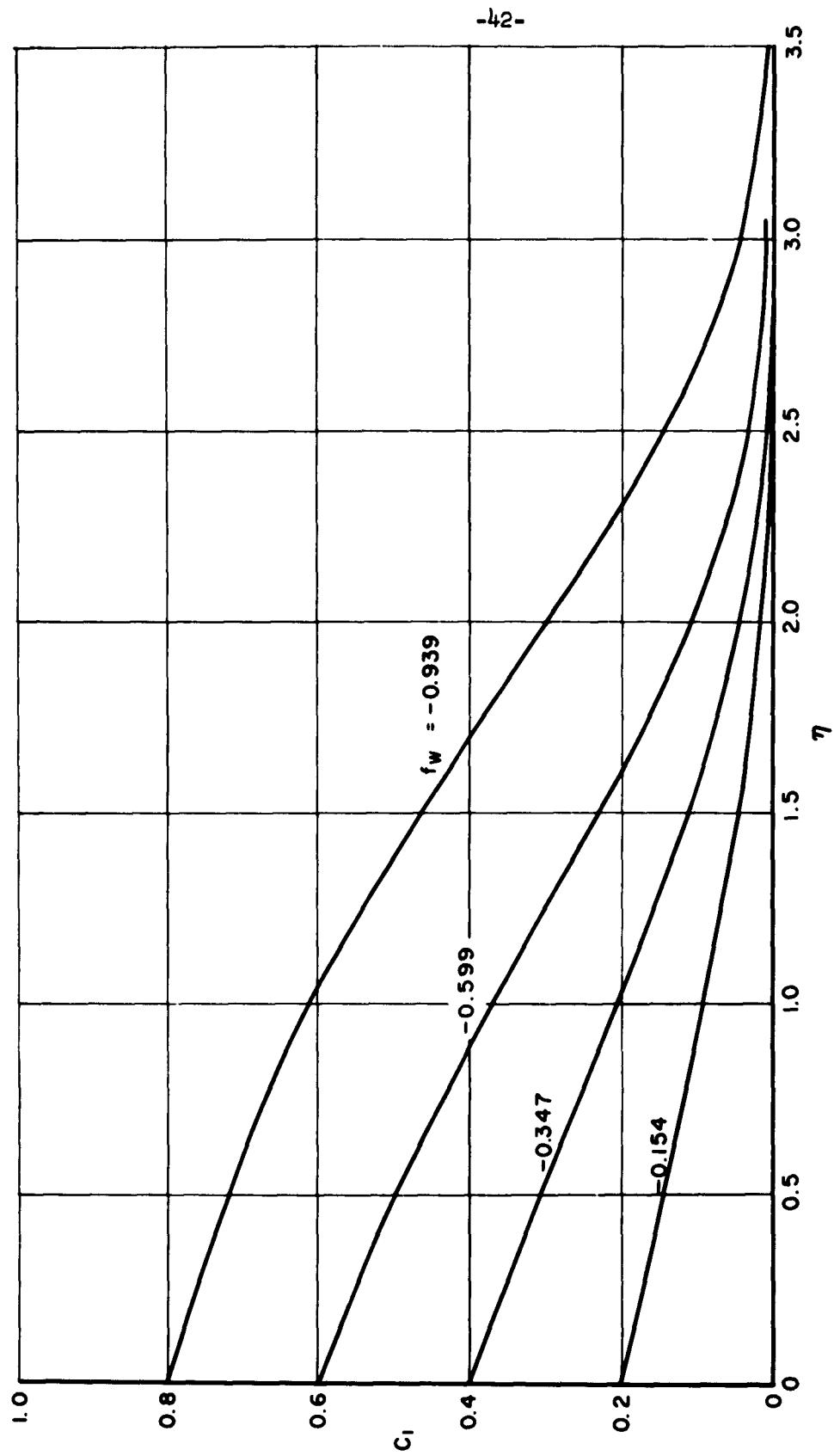


Fig. 5 — Concentration profiles: carbon dioxide

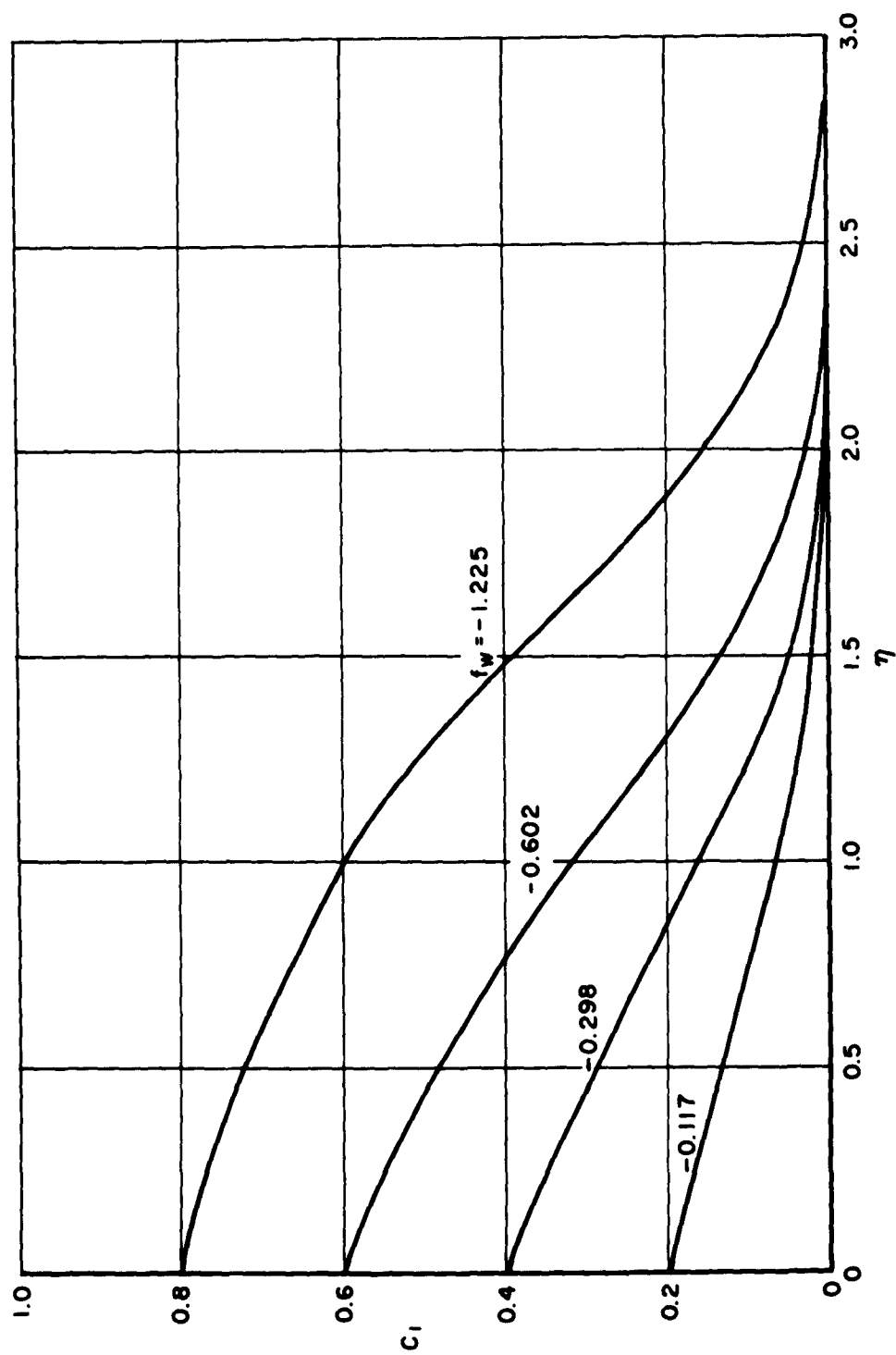


Fig. 6—Concentration profiles: iodine

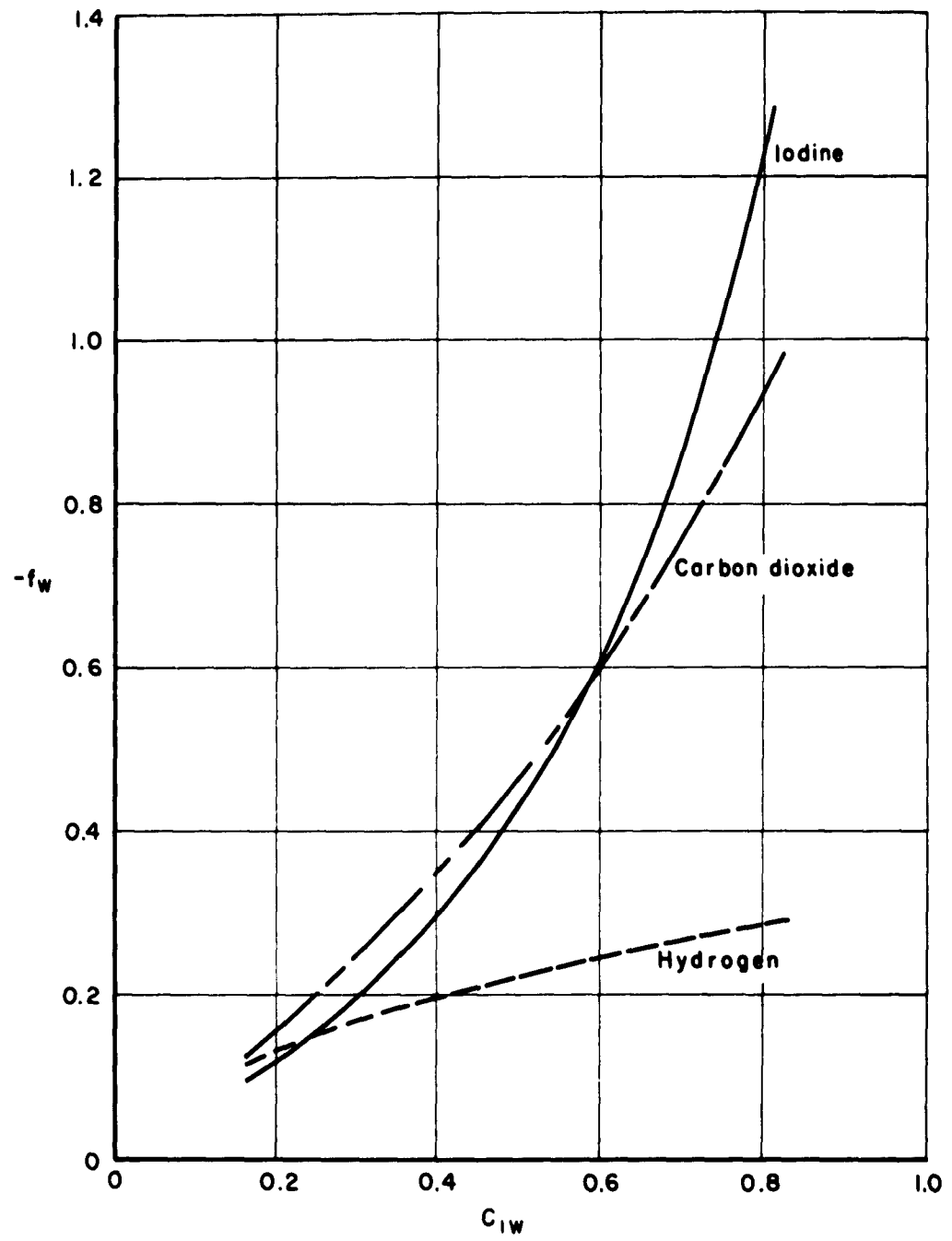


Fig. 7— Injection rate vs wall concentration

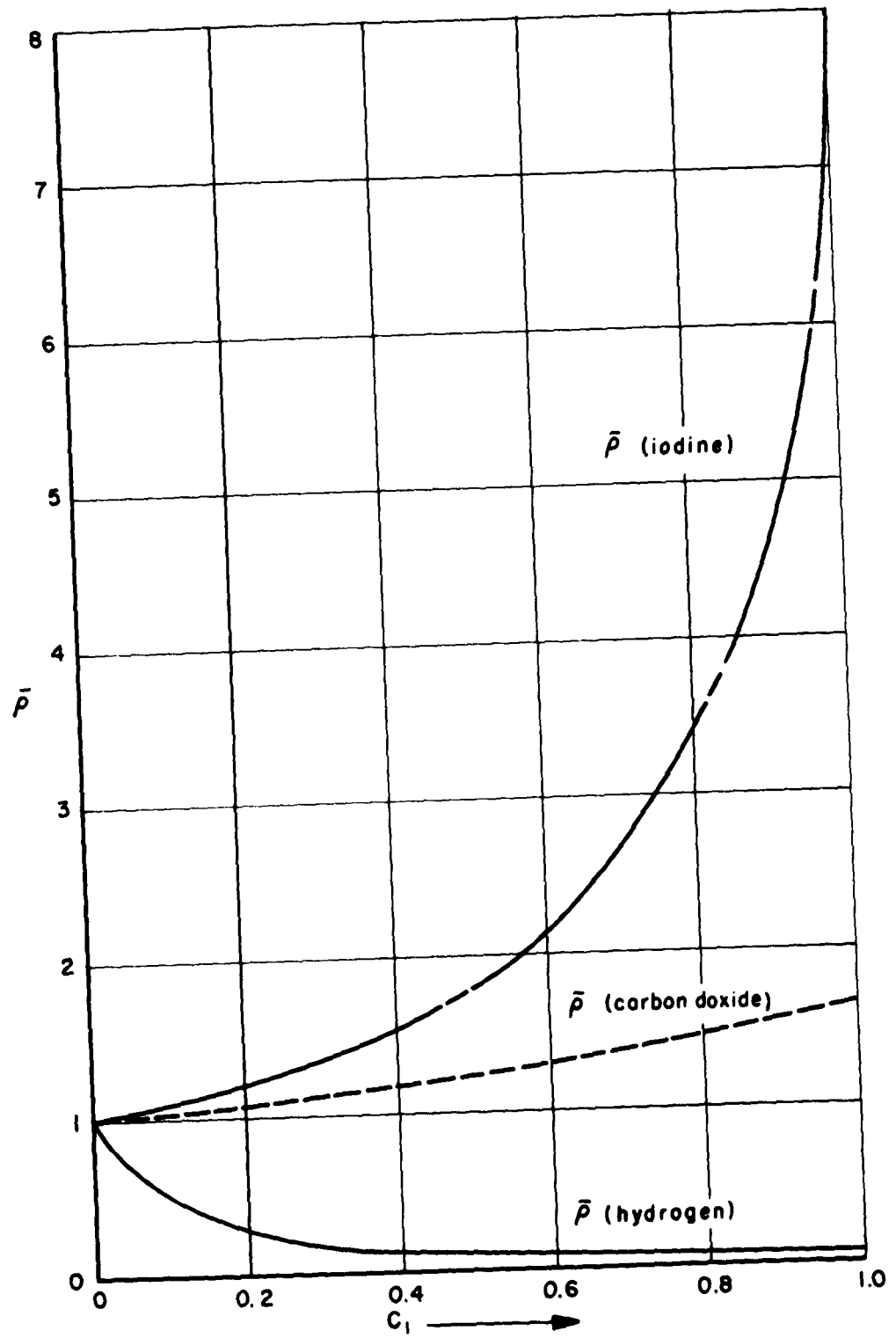


Fig. 8 —  $\bar{p}$  vs  $C_1$

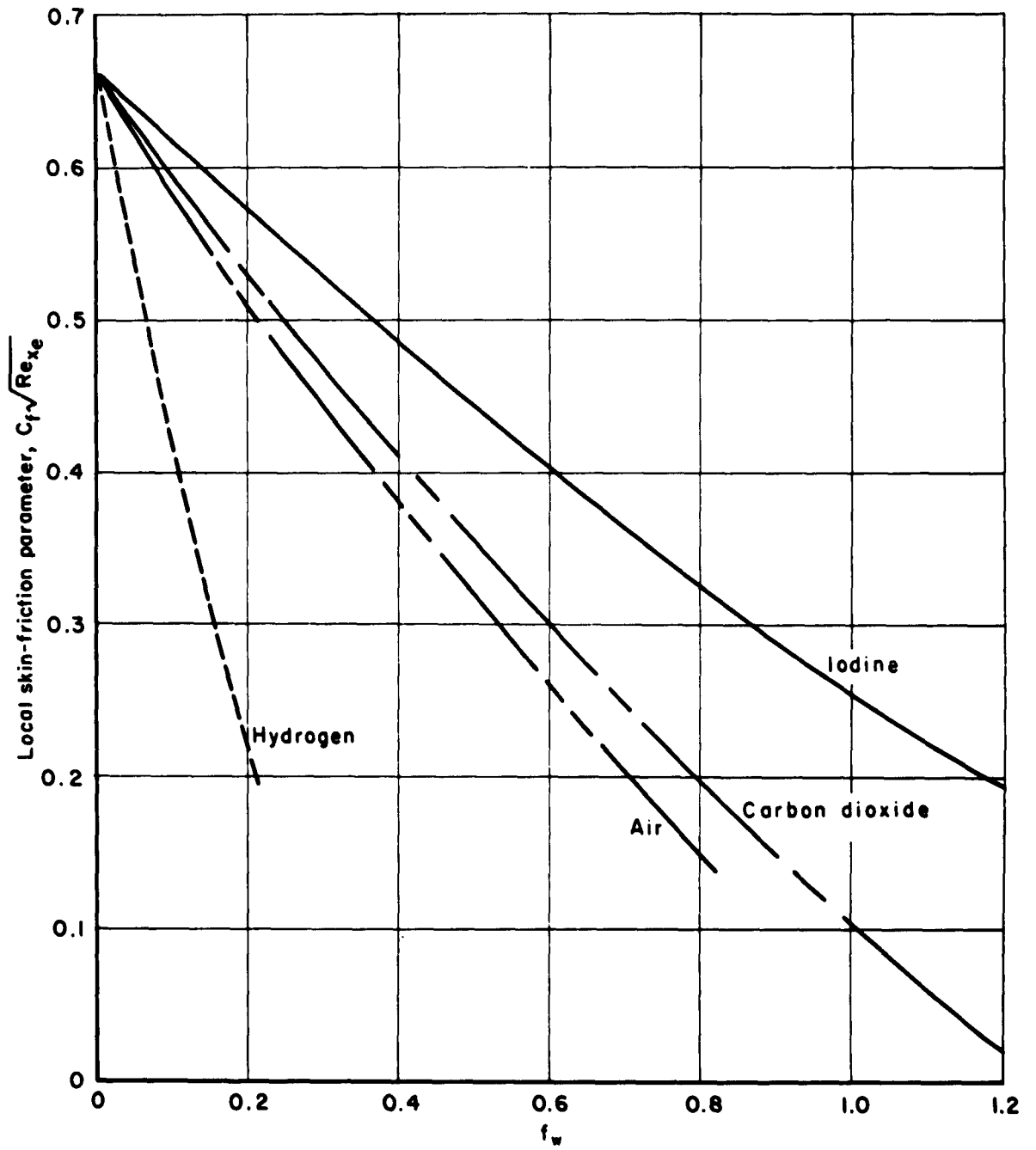


Fig. 9—Friction coefficient vs injection rate

friction coefficient is zero, extending the hydrogen curve shows that at  $f_w = 0.35$  the hydrogen would have blown the boundary layer off the surface; in the case of carbon dioxide or iodine it is still firmly entrenched. This indicates the destabilizing character of hydrogen injection. Both carbon dioxide and iodine increase the friction coefficient over that value which it would have with air injection. At low concentrations this increase is seen to be small. Carbon dioxide, particularly, gives values which differ by a small, almost constant factor from the normal air values. In the case of iodine, the breach becomes geometrically greater as the injection rate increases. The drag coefficient is plotted against wall concentration in Fig. 10.

The next series of graphs (Figs. 11 - 13) shows the "stability" function,  $\bar{\rho} \frac{d}{d\eta} \left[ \frac{u}{u_e} \right]$  plotted against  $\eta$  for varying wall concentrations. As was stated previously, the characteristic hump indicative of a point of inflection shows that the system is unstable. On the other hand, it is to be noted that this criterion applies to an incompressible fluid and that ours is not truly incompressible, since the density and viscosity vary throughout the boundary layer. The variation of these two parameters is dependent upon concentration and not upon temperature. Therefore, any conclusions which can be drawn are at best qualitative and are intended to show direction rather than order of magnitude. Since for large enough  $\eta$  all the stability curves must approach zero, a hump will automatically appear if the slope of the function at the wall is greater than zero. In the case of hydrogen it is immediately apparent that instability exists for all conditions. For concentrations of 0.2 and 0.4 the humps are still apparent, but for higher concentrations they appear in the boundary layer



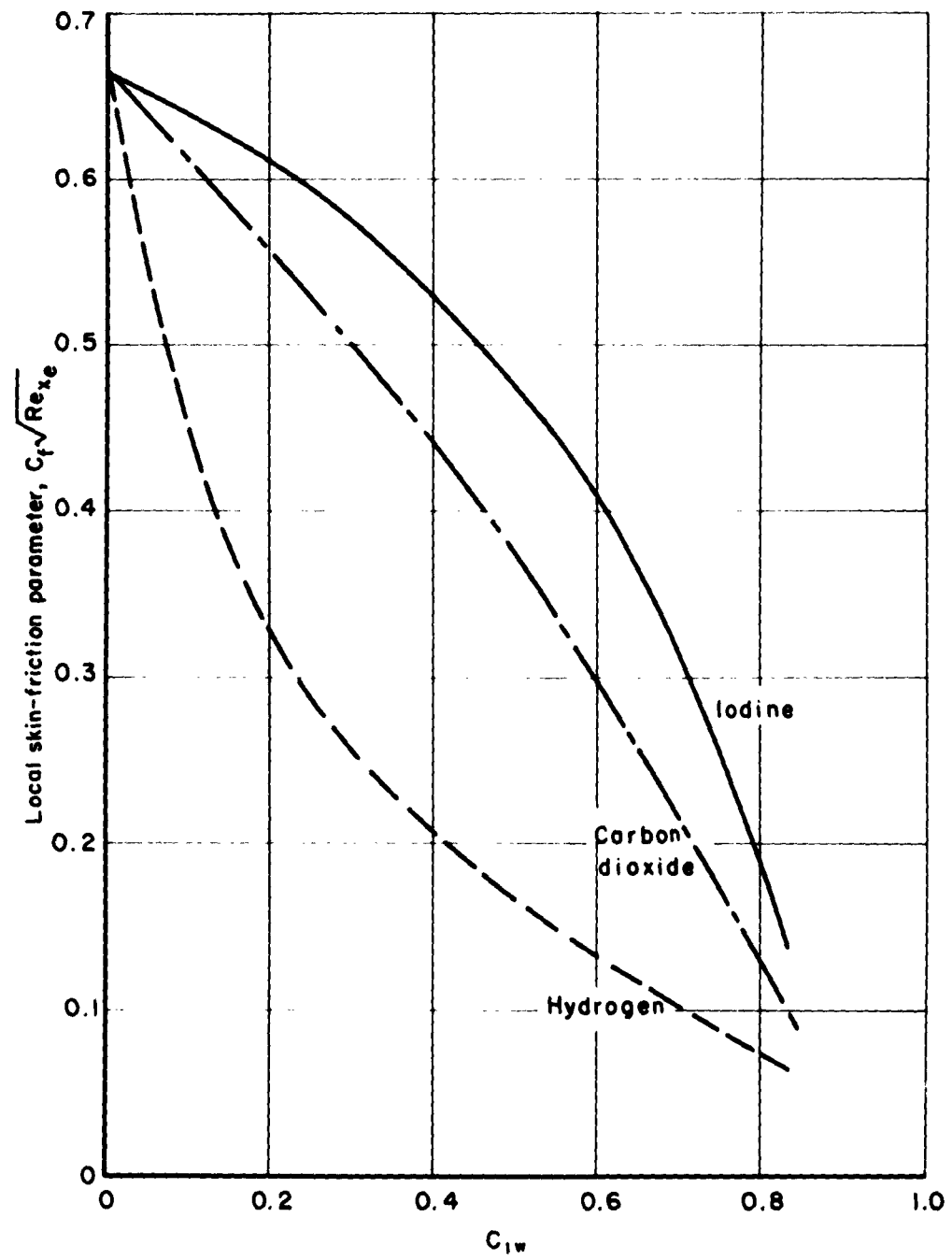


Fig. 10— Friction coefficient vs injection composition

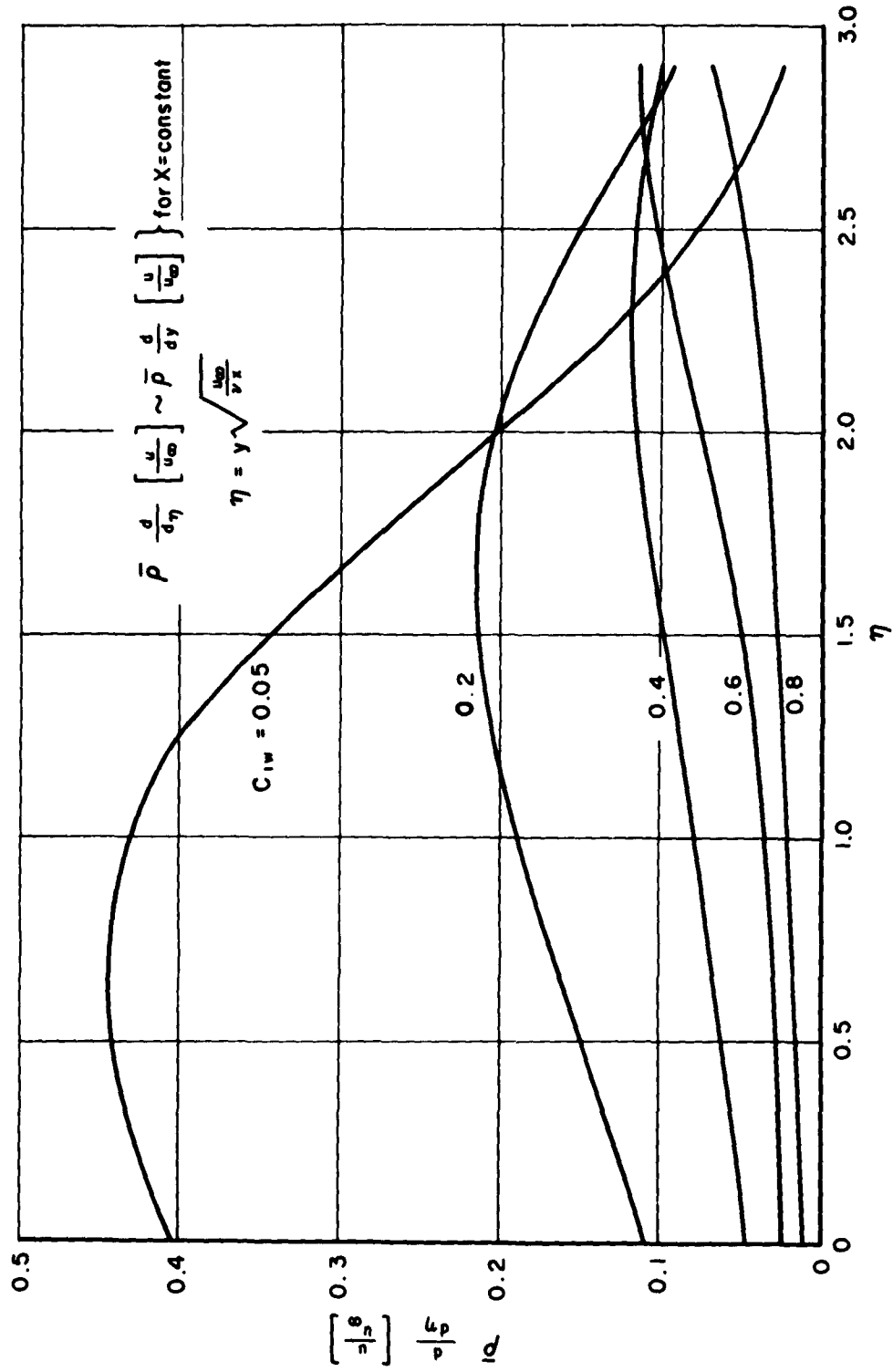


Fig. 11 —  $\bar{p} \frac{d}{d\eta} \left[ \frac{u}{u_\infty} \right]$  vs  $\eta$ : hydrogen

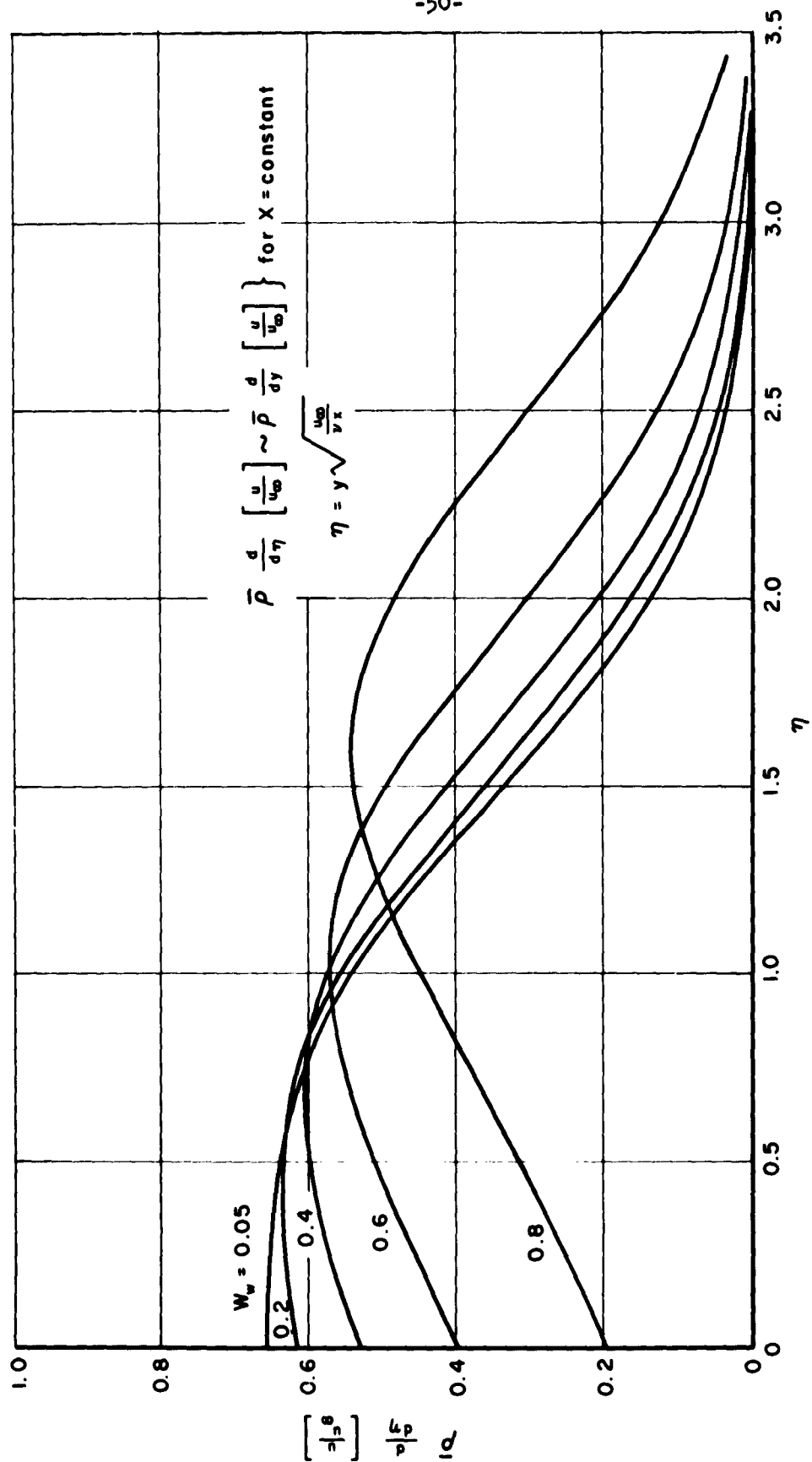


Fig. 12 —  $\bar{p} \frac{d}{d\eta} \left[ \frac{u}{u_{\infty}} \right]$  vs  $\eta$ ; carbon dioxide

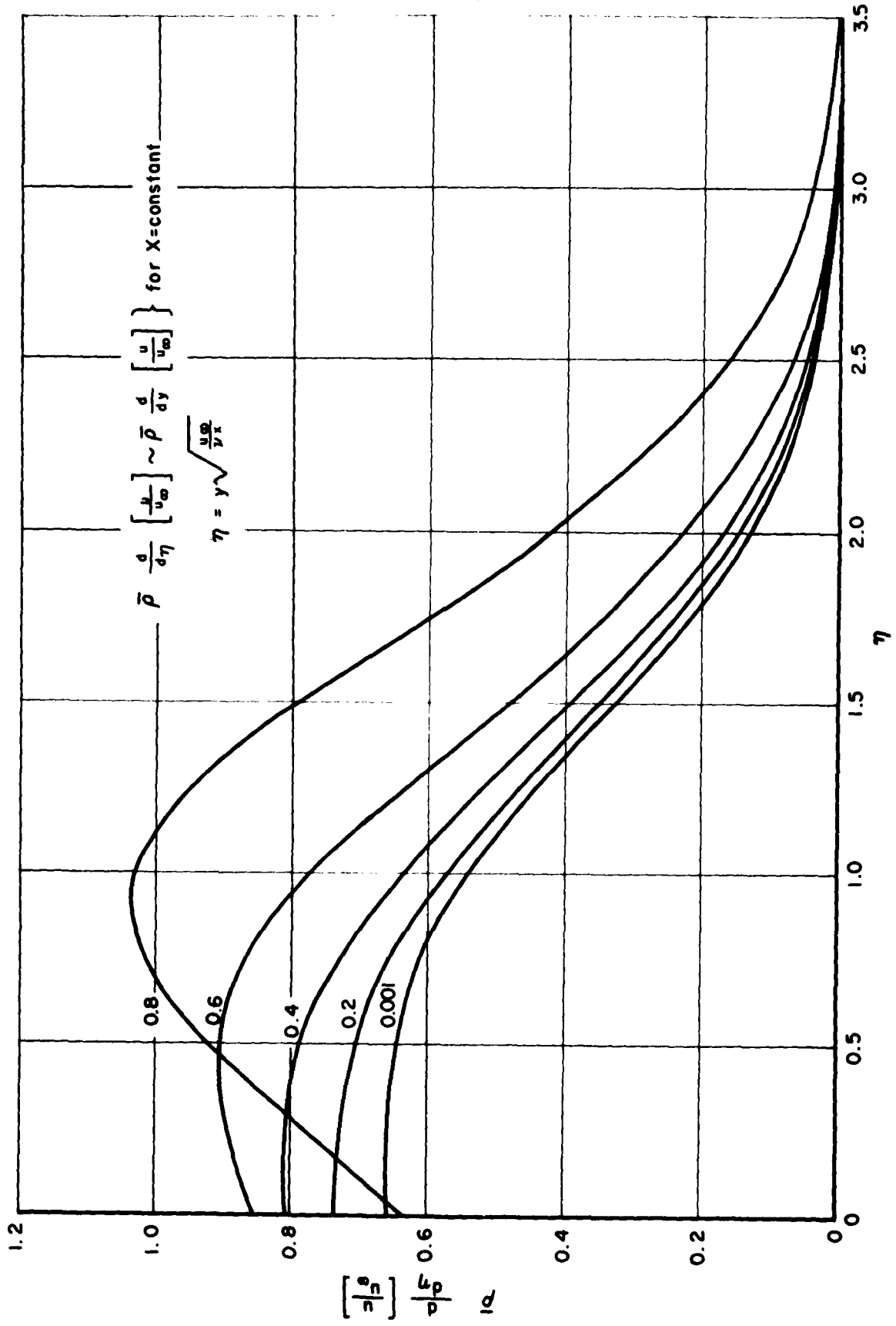


Fig. 13 —  $\bar{p} \frac{d}{d\eta} \left[ \frac{u}{u_\infty} \right]$  vs  $\eta$ : iodine

for  $\eta > 3$ . In the case of carbon dioxide, it can be seen that the inflection points have moved toward the left; for a concentration of 0.05 the inflection hump has almost disappeared. In the case of iodine, the inflection points are close to the left axis and are no longer discernible for the three lower concentrations. Even for a concentration of 0.2, calculations show that stability has been achieved. For lower concentration it is evident that these curves have flattened out and that they show a stable character. A careful numerical analysis should indicate the most advantageous concentration. All three materials tend toward the Blasius profile in the limit (as concentration approaches zero). At large concentrations all are obviously unstable. It will be in the range of small injection rates that the difference in the three materials will be most noticeable. This would actually be the case if an experimental model were to be made. It goes without saying that a careful numerical analysis of the lower concentrations is in order.

The critical Reynolds numbers are plotted in Fig. 14. Here it is again evident that hydrogen injection reduces the stability character of a boundary layer significantly, even at low concentrations. Iodine, of the three materials, gives the least reduction and is the best injection medium according to stability considerations. A general direction of comparative stability characteristics has been shown.

Injection media, such as iodine, that are less destabilizing prevent immediate transition and increase the area affected by the low-drag laminar flow. However, the medium itself produces an increase of drag (above that obtained under similar conditions with hydrogen injection) so that some of its advantage is negated. The wall concentration at which iodine most

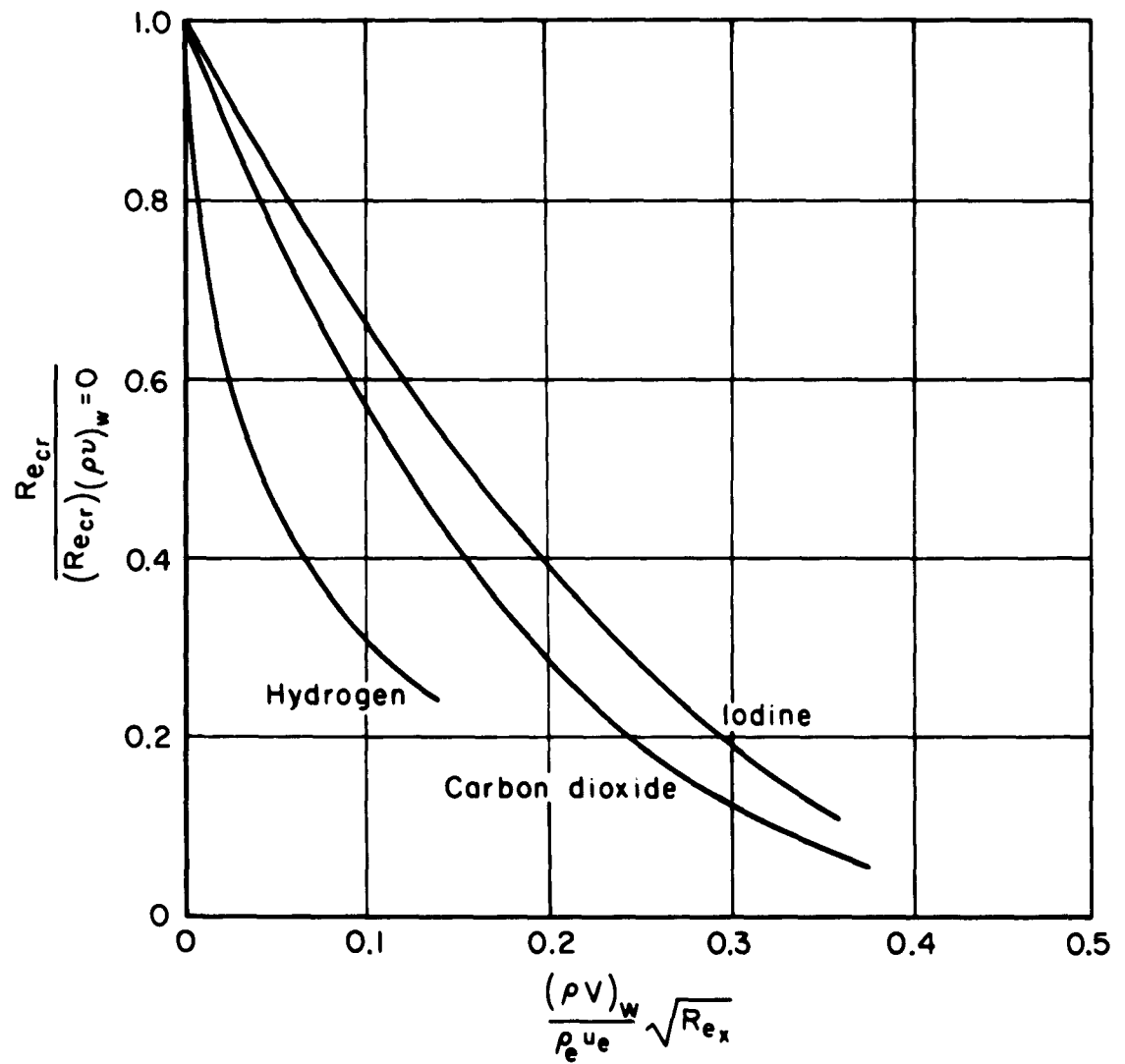


Fig. 14—The influence of mass transfer on the stability of the incompressible laminar boundary layer

effectively prolongs laminar flow must first be obtained. The critical Reynolds number for this condition may then be found, and a comparative estimate may be made between turbulent, impermeable-flat-plate, and iodine-air boundary layers.

Appendix A

SOLUTION OF THE LAMINAR BOUNDARY LAYER



Table 2

SOLUTION OF THE LAMINAR BOUNDARY LAYER WITH  
CARBON DIOXIDE INJECTION

| $\eta$                     | $\frac{C_{1W} - C_1}{C_{1W}}$ | $\frac{U}{U_e}$ | $y_0$   | $y_2$  | $y_3 \times 10^4$ | $y_4 \times 10^4$ |
|----------------------------|-------------------------------|-----------------|---------|--------|-------------------|-------------------|
| (C <sub>1W</sub> = 0.0001) |                               |                 |         |        |                   |                   |
| 0                          | 0                             | 0               | -0.0001 | 1.3281 | 1.0               | -0.64979          |
| 0.2                        | 0.1299                        | 0.1328          | 0.0265  | 1.3258 | 0.87010           | -0.64872          |
| 0.4                        | 0.2590                        | 0.2647          | 0.1060  | 1.3095 | 0.74095           | -0.64122          |
| 0.6                        | 0.3856                        | 0.3938          | 0.2379  | 1.2663 | 0.61445           | -0.62132          |
| 0.8                        | 0.5064                        | 0.5167          | 0.4202  | 1.1866 | 0.49356           | -0.58450          |
| 1.0                        | 0.6181                        | 0.6298          | 0.6499  | 1.0670 | 0.38191           | -0.52894          |
| 1.2                        | 0.7169                        | 0.7290          | 0.9222  | 0.9124 | 0.28311           | -0.45656          |
| 1.4                        | 0.8000                        | 0.8115          | 1.2309  | 0.7360 | 0.20003           | -0.37310          |
| 1.6                        | 0.8659                        | 0.8761          | 1.5690  | 0.5565 | 0.13406           | -0.28688          |
| 1.8                        | 0.9151                        | 0.9233          | 1.9294  | 0.3924 | 0.08488           | -0.20654          |
| 2.0                        | 0.9494                        | 0.9555          | 2.3056  | 0.2570 | 0.05060           | -0.13874          |
| 2.2                        | 0.9717                        | 0.9759          | 2.6923  | 0.1559 | 0.02832           | -0.8674           |
| 2.4                        | 0.9851                        | 0.9878          | 3.0852  | 0.0875 | 0.01486           | -0.05040          |
| 2.6                        | 0.9927                        | 0.9942          | 3.4818  | 0.0454 | 0.00729           | -0.02718          |
| 2.8                        | 0.9967                        | 0.9975          | 3.8802  | 0.0217 | 0.00334           | -0.01361          |
| 2.9                        | 0.9978                        | 0.9989          | 4.0798  | 0.0146 | 0.00220           | -0.00936          |
| (C <sub>1W</sub> = 0.001)  |                               |                 |         |        |                   |                   |
| 0                          | 0                             | 0               | -0.0007 | 1.3273 | 10.0              | -6.4950           |
| 0.2                        | 0.1298                        | 0.1327          | 0.0259  | 1.3251 | 8.7018            | -6.4851           |
| 0.4                        | 0.2589                        | 0.2646          | 0.1054  | 1.3090 | 7.4110            | -6.4109           |
| 0.6                        | 0.3854                        | 0.3936          | 0.2372  | 1.2660 | 6.1464            | -6.2128           |
| 0.8                        | 0.5062                        | 0.5166          | 0.4195  | 1.1865 | 4.9378            | -5.8454           |
| 1.0                        | 0.6179                        | 0.6296          | 0.6492  | 1.0670 | 3.8212            | -5.2905           |
| 1.2                        | 0.7167                        | 0.7288          | 0.9214  | 0.9125 | 2.8332            | -4.5673           |
| 1.4                        | 0.7998                        | 0.8114          | 1.2301  | 0.7363 | 2.0020            | -3.7330           |
| 1.6                        | 0.8658                        | 0.8760          | 1.5682  | 0.5568 | 1.3419            | -2.8708           |
| 1.8                        | 0.9150                        | 0.9232          | 1.9286  | 0.3926 | 0.8498            | -2.0672           |
| 2.0                        | 0.9493                        | 0.9555          | 2.3048  | 0.2572 | 0.5066            | -1.3888           |
| 2.2                        | 0.9716                        | 0.9758          | 2.6914  | 0.1560 | 0.2836            | -0.8684           |
| 2.4                        | 0.9851                        | 0.9878          | 3.0844  | 0.0876 | 0.1488            | -0.5046           |
| 2.6                        | 0.9927                        | 0.9942          | 3.4809  | 0.0454 | 0.0730            | -0.2723           |
| 2.8                        | 0.9966                        | 0.9975          | 3.8793  | 0.0218 | 0.0334            | -0.1363           |
| 2.9                        | 0.9978                        | 0.9984          | 4.0789  | 0.0146 | 0.0221            | -0.0938           |

Table 2 (Contd.)

| $\eta$                    | $\frac{c_{1W} - c_1}{c_{1W}}$ | $\frac{U}{U_e}$ | $y_0$   | $y_2$  | $y_3 \times 10^3$ | $y_4 \times 10^3$ |
|---------------------------|-------------------------------|-----------------|---------|--------|-------------------|-------------------|
| (c <sub>1W</sub> = 0.005) |                               |                 |         |        |                   |                   |
| 0                         | 0                             | 0               | -0.0035 | 1.3236 | 5.0               | -3.2411           |
| 0.2                       | 0.1294                        | 0.1324          | 0.0231  | 1.3222 | 4.3529            | -3.2378           |
| 0.4                       | 0.2582                        | 0.2641          | 0.1025  | 1.3068 | 3.7089            | -3.2025           |
| 0.6                       | 0.3845                        | 0.3930          | 0.2342  | 1.2646 | 3.0776            | -3.1054           |
| 0.8                       | 0.5052                        | 0.5159          | 0.4164  | 1.1859 | 2.4738            | -2.9235           |
| 1.0                       | 0.6169                        | 0.6289          | 0.6460  | 1.0672 | 1.9155            | -2.6477           |
| 1.2                       | 0.7158                        | 0.7282          | 0.9180  | 0.9132 | 1.4210            | -2.2874           |
| 1.4                       | 0.7990                        | 0.8108          | 1.2265  | 0.7373 | 1.0043            | -1.8709           |
| 1.6                       | 0.8652                        | 0.8755          | 1.5645  | 0.5580 | 0.6739            | -1.2327           |
| 1.8                       | 0.9146                        | 0.9229          | 1.9248  | 0.3937 | 0.4270            | -1.0376           |
| 2.0                       | 0.9490                        | 0.9552          | 2.3010  | 0.2581 | 0.2548            | -0.6976           |
| 2.2                       | 0.9714                        | 0.9757          | 2.6875  | 0.1567 | 0.1427            | -0.4366           |
| 2.4                       | 0.9850                        | 0.9877          | 3.0804  | 0.0880 | 0.0749            | -0.2539           |
| 2.6                       | 0.9926                        | 0.9942          | 3.4770  | 0.0457 | 0.0368            | -0.1371           |
| 2.8                       | 0.9966                        | 0.9974          | 3.8754  | 0.0219 | 0.0169            | -0.0687           |
| 2.9                       | 0.9978                        | 0.9984          | 4.0750  | 0.0147 | 0.0111            | -0.0473           |
| (c <sub>1W</sub> = 0.01)  |                               |                 |         |        |                   |                   |
| 0                         | 0                             | 0               | -0.0069 | 1.3190 | 10.0              | -6.4661           |
| 0.2                       | 0.1289                        | 0.1321          | 0.0195  | 1.3184 | 8.7106            | -6.4638           |
| 0.4                       | 0.2574                        | 0.2634          | 0.0989  | 1.3041 | 7.4264            | -6.3976           |
| 0.6                       | 0.3834                        | 0.3922          | 0.2305  | 1.2629 | 6.1663            | -6.2080           |
| 0.8                       | 0.5040                        | 0.5150          | 0.4125  | 1.1851 | 4.9598            | -5.8491           |
| 1.0                       | 0.6157                        | 0.6280          | 0.6419  | 1.0673 | 3.8433            | -5.3017           |
| 1.2                       | 0.7147                        | 0.7273          | 0.9137  | 0.9141 | 2.8534            | -4.5840           |
| 1.4                       | 0.7981                        | 0.8101          | 1.2221  | 0.7387 | 2.0191            | -3.7528           |
| 1.6                       | 0.8645                        | 0.8750          | 1.5599  | 0.5595 | 1.3554            | -2.8908           |
| 1.8                       | 0.9140                        | 0.9225          | 1.9201  | 0.3952 | 0.8596            | -2.0852           |
| 2.0                       | 0.9487                        | 0.9550          | 2.2961  | 0.2593 | 0.5132            | -1.4033           |
| 2.2                       | 0.9712                        | 0.9755          | 2.6826  | 0.1576 | 0.2878            | -0.8790           |
| 2.4                       | 0.9849                        | 0.9876          | 3.0755  | 0.0886 | 0.1512            | -0.5116           |
| 2.6                       | 0.9926                        | 0.9941          | 3.4720  | 0.0460 | 0.0743            | -0.2765           |
| 2.8                       | 0.9966                        | 0.9974          | 3.8704  | 0.0221 | 0.0341            | -0.1387           |
| 2.9                       | 0.9977                        | 0.9983          | 4.0700  | 0.0148 | 0.0225            | -0.0955           |

Table 2 (Contd.)

| $\eta$                   | $\frac{C_{1W} - C_1}{C_{1W}}$ | $\frac{U}{U_e}$ | $y_0$   | $y_2$  | $y_3$    | $y_4$     |
|--------------------------|-------------------------------|-----------------|---------|--------|----------|-----------|
| (C <sub>1W</sub> = 0.05) |                               |                 |         |        |          |           |
| 0                        | 0                             | 0               | -0.0355 | 1.2810 | 0.050    | -0.031670 |
| 0.2                      | 0.1250                        | 0.1292          | -0.0092 | 1.2879 | 0.043749 | -0.031827 |
| 0.4                      | 0.2504                        | 0.2584          | 0.0693  | 1.2813 | 0.037479 | -0.031676 |
| 0.6                      | 0.3744                        | 0.3856          | 0.1998  | 1.2482 | 0.031281 | -0.030918 |
| 0.8                      | 0.4940                        | 0.5076          | 0.3804  | 1.1786 | 0.025301 | -0.029314 |
| 1.0                      | 0.6056                        | 0.6206          | 0.6081  | 1.0682 | 0.019721 | -0.026751 |
| 1.2                      | 0.7053                        | 0.7205          | 0.8784  | 0.9210 | 0.014733 | -0.023296 |
| 1.4                      | 0.7901                        | 0.8042          | 1.1852  | 0.7495 | 0.010493 | -0.019214 |
| 1.6                      | 0.8582                        | 0.8704          | 1.5218  | 0.5719 | 0.007090 | -0.014914 |
| 1.8                      | 0.9094                        | 0.9192          | 1.8809  | 0.4070 | 0.004527 | -0.010842 |
| 2.0                      | 0.9456                        | 0.9527          | 2.2562  | 0.2691 | 0.002722 | -0.007554 |
| 2.2                      | 0.9693                        | 0.9741          | 2.6422  | 0.1649 | 0.001537 | -0.004642 |
| 2.4                      | 0.9837                        | 0.9868          | 3.0348  | 0.0935 | 0.000813 | -0.002723 |
| 2.6                      | 0.9920                        | 0.9937          | 3.4311  | 0.0490 | 0.000402 | -0.001483 |
| 2.8                      | 0.9963                        | 0.9972          | 3.8294  | 0.0237 | 0.000186 | -0.000750 |
| 2.9                      | 0.9975                        | 0.9982          | 4.0290  | 0.0160 | 0.000123 | -0.000518 |
| (C <sub>1W</sub> = 0.2)  |                               |                 |         |        |          |           |
| 0                        | 0                             | 0               | -0.1539 | 1.1264 | 0.2000   | -0.1157   |
| 0.2                      | 0.1097                        | 0.1168          | -0.1291 | 1.1604 | 0.1780   | -0.1187   |
| 0.4                      | 0.2228                        | 0.2363          | -0.0543 | 1.1833 | 0.1554   | -0.1208   |
| 0.6                      | 0.3378                        | 0.3567          | 0.0704  | 1.1824 | 0.1324   | -0.1207   |
| 0.8                      | 0.4522                        | 0.4750          | 0.2440  | 1.1463 | 0.1096   | -0.1174   |
| 1.0                      | 0.5625                        | 0.5873          | 0.4641  | 1.0679 | 0.0875   | -0.1102   |
| 1.2                      | 0.6646                        | 0.6893          | 0.7267  | 0.9477 | 0.0671   | -0.0988   |
| 1.4                      | 0.7545                        | 0.7772          | 1.0264  | 0.7949 | 0.0491   | -0.0841   |
| 1.6                      | 0.8293                        | 0.8486          | 1.3568  | 0.6260 | 0.0342   | -0.0675   |
| 1.8                      | 0.8878                        | 0.9030          | 1.7110  | 0.4603 | 0.0224   | -0.0507   |
| 2.0                      | 0.9305                        | 0.9416          | 2.0827  | 0.3148 | 0.0139   | -0.0356   |
| 2.2                      | 0.9596                        | 0.9670          | 2.4662  | 0.1997 | 0.0081   | -0.0232   |
| 2.4                      | 0.9780                        | 0.9826          | 2.8573  | 0.1172 | 0.0044   | -0.0141   |
| 2.6                      | 0.9888                        | 0.9915          | 3.2527  | 0.0636 | 0.0022   | -0.0080   |
| 2.8                      | 0.9946                        | 0.9961          | 3.6506  | 0.0319 | 0.0011   | -0.0042   |
| 2.9                      | 0.9964                        | 0.9974          | 3.8500  | 0.0219 | 0.0007   | -0.0029   |

Table 2 (Contd.)

| $\eta$           | $\frac{c_{1w} - c_1}{c_{1w}}$ | $\frac{U}{U_e}$ | $y_0$   | $y_2$  | $y_3$  | $y_4$   |
|------------------|-------------------------------|-----------------|---------|--------|--------|---------|
| $(c_{1w} = 0.4)$ |                               |                 |         |        |        |         |
| 0                | 0                             | 0               | -0.3474 | 0.8850 | 0.4000 | -0.1960 |
| 0.2              | 0.0876                        | 0.0959          | -0.3257 | 0.9502 | 0.3649 | -0.2072 |
| 0.4              | 0.1813                        | 0.1979          | -0.2593 | 1.0106 | 0.3275 | -0.2176 |
| 0.6              | 0.2806                        | 0.3050          | -0.1472 | 1.0549 | 0.2878 | -0.2253 |
| 0.8              | 0.3838                        | 0.4148          | 0.0112  | 1.0706 | 0.2465 | -0.2238 |
| 1.0              | 0.4884                        | 0.5238          | 0.2146  | 1.0470 | 0.2046 | -0.2238 |
| 1.2              | 0.5906                        | 0.6275          | 0.4605  | 0.9779 | 0.1637 | -0.2110 |
| 1.4              | 0.6861                        | 0.7215          | 0.7446  | 0.8658 | 0.1255 | -0.1895 |
| 1.6              | 0.7708                        | 0.8020          | 1.0615  | 0.7216 | 0.0917 | -0.1610 |
| 1.8              | 0.8413                        | 0.8668          | 1.4048  | 0.5629 | 0.0635 | -0.1285 |
| 2.0              | 0.8963                        | 0.9156          | 1.7682  | 0.4092 | 0.0415 | -0.0959 |
| 2.2              | 0.9363                        | 0.9498          | 2.1460  | 0.2764 | 0.0255 | -0.0666 |
| 2.4              | 0.9633                        | 0.9720          | 2.5333  | 0.1729 | 0.0147 | -0.0431 |
| 2.6              | 0.9802                        | 0.9854          | 2.9265  | 0.1001 | 0.0079 | -0.0258 |
| 2.8              | 0.9900                        | 0.9930          | 3.3231  | 0.0536 | 0.0040 | -0.0144 |
| 2.9              | 0.9931                        | 0.9952          | 3.5222  | 0.0380 | 0.0028 | -0.0104 |
| $(c_{1w} = 0.6)$ |                               |                 |         |        |        |         |
| 0                | 0                             | 0               | -0.5990 | 0.5955 | 0.6000 | -0.2252 |
| 0.2              | 0.0630                        | 0.0681          | -0.5824 | 0.6762 | 0.5622 | -0.2563 |
| 0.4              | 0.1329                        | 0.1447          | -0.5306 | 0.7619 | 0.5202 | -0.2683 |
| 0.6              | 0.2102                        | 0.2299          | -0.4408 | 0.8452 | 0.4739 | -0.2896 |
| 0.8              | 0.2946                        | 0.3228          | -0.3106 | 0.9154 | 0.4232 | -0.3076 |
| 1.0              | 0.3852                        | 0.4212          | -0.1388 | 0.9602 | 0.3689 | -0.3192 |
| 1.2              | 0.4798                        | 0.5218          | 0.0745  | 0.9674 | 0.3689 | -0.3192 |
| 1.4              | 0.5752                        | 0.6201          | 0.3275  | 0.9287 | 0.2549 | -0.3105 |
| 1.6              | 0.6673                        | 0.7113          | 0.6164  | 0.8435 | 0.1996 | -0.2862 |
| 1.8              | 0.7516                        | 0.7913          | 0.9362  | 0.7204 | 0.1491 | -0.2496 |
| 2.0              | 0.8241                        | 0.8571          | 1.2813  | 0.5756 | 0.1055 | -0.2045 |
| 2.2              | 0.8825                        | 0.9078          | 1.6456  | 0.4285 | 0.0705 | -0.1566 |
| 2.4              | 0.9262                        | 0.9440          | 2.0237  | 0.2962 | 0.0443 | -0.1117 |
| 2.6              | 0.9565                        | 0.9682          | 2.4111  | 0.1898 | 0.0261 | -0.0740 |
| 2.8              | 0.9760                        | 0.9831          | 2.8043  | 0.1126 | 0.0144 | -0.0455 |
| 2.9              | 0.9826                        | 0.9880          | 3.0022  | 0.0842 | 0.0104 | -0.0346 |

Table 2 (Contd.)

| $\eta$                  | $\frac{c_{1W} - c_1}{c_{1W}}$ | $\frac{U}{U_e}$ | $y_0$   | $y_2$  | $y_3$  | $y_4$   |
|-------------------------|-------------------------------|-----------------|---------|--------|--------|---------|
| (c <sub>1W</sub> = 0.8) |                               |                 |         |        |        |         |
| 0                       | 0                             | 0               | -0.9393 | 0.2607 | 0.8000 | -0.1766 |
| 0.2                     | 0.0345                        | 0.0319          | -0.9309 | 0.3205 | 0.7724 | -0.2009 |
| 0.4                     | 0.0742                        | 0.0709          | -0.9036 | 0.3921 | 0.7406 | -0.2283 |
| 0.6                     | 0.1201                        | 0.1182          | -0.8538 | 0.4753 | 0.7039 | -0.2587 |
| 0.8                     | 0.1728                        | 0.1748          | -0.7779 | 0.5678 | 0.6618 | -0.2911 |
| 1.0                     | 0.2329                        | 0.2412          | -0.6720 | 0.6644 | 0.6136 | -0.3239 |
| 1.2                     | 0.3009                        | 0.3175          | -0.5327 | 0.7565 | 0.5593 | -0.3548 |
| 1.4                     | 0.3763                        | 0.4022          | -0.3574 | 0.8322 | 0.4990 | -0.3801 |
| 1.6                     | 0.4580                        | 0.4928          | -0.1448 | 0.8781 | 0.4336 | -0.3957 |
| 1.8                     | 0.5440                        | 0.5854          | 0.1048  | 0.8824 | 0.3648 | -0.3971 |
| 2.0                     | 0.6306                        | 0.6752          | 0.3890  | 0.8388 | 0.2955 | -0.3812 |
| 2.2                     | 0.7139                        | 0.7573          | 0.7039  | 0.7499 | 0.2289 | -0.3473 |
| 2.4                     | 0.7893                        | 0.8280          | 1.0444  | 0.6275 | 0.1685 | -0.2980 |
| 2.6                     | 0.8533                        | 0.8847          | 1.4048  | 0.4894 | 0.1173 | -0.2394 |
| 2.8                     | 0.9039                        | 0.9273          | 1.7800  | 0.3548 | 0.0769 | -0.1792 |
| 2.9                     | 0.9240                        | 0.9436          | 1.9716  | 0.2937 | 0.0608 | -0.1509 |

Table 3

SOLUTION OF THE LAMINAR BOUNDARY LAYER WITH  
HYDROGEN INJECTION

| $\eta$                     | $\frac{c_{1W} - c_1}{c_{1W}}$ | $\frac{U}{U_e}$ | $y_0$   | $y_2$  | $y_3 \times 10^4$ | $y_4 \times 10^4$ |
|----------------------------|-------------------------------|-----------------|---------|--------|-------------------|-------------------|
| (c <sub>1W</sub> = 0.0001) |                               |                 |         |        |                   |                   |
| 0                          | 0                             | 0               | -0.0002 | 1.3278 | 1.0               | -0.39761          |
| 0.2                        | 0.0796                        | 0.1327          | 0.0263  | 1.3250 | 0.92088           | -0.39748          |
| 0.4                        | 0.1591                        | 0.2645          | 0.1058  | 1.3088 | 0.84087           | -0.39639          |
| 0.6                        | 0.2382                        | 0.3933          | 0.2374  | 1.2657 | 0.76176           | -0.39347          |
| 0.8                        | 0.3165                        | 0.5165          | 0.4194  | 1.1862 | 0.68350           | -0.38786          |
| 1.0                        | 0.3933                        | 0.6295          | 0.6488  | 1.0669 | 0.60670           | -0.37887          |
| 1.2                        | 0.4679                        | 0.7287          | 0.9208  | 0.9125 | 0.53209           | -0.36600          |
| 1.4                        | 0.5395                        | 0.8112          | 1.2291  | 0.7364 | 0.46047           | -0.34904          |
| 1.6                        | 0.6074                        | 0.8758          | 1.5670  | 0.5570 | 0.39265           | -0.32815          |
| 1.8                        | 0.6706                        | 0.9231          | 1.9271  | 0.3929 | 0.32937           | -0.30377          |
| 2.0                        | 0.7287                        | 0.9554          | 2.3031  | 0.2574 | 0.27127           | -0.27666          |
| 2.2                        | 0.7812                        | 0.9758          | 2.6896  | 0.1562 | 0.21879           | 0.24776           |
| 2.4                        | 0.8278                        | 0.9877          | 3.0824  | 0.0877 | 0.17219           | -0.21808          |
| 2.6                        | 0.8685                        | 0.9942          | 3.4788  | 0.0455 | 0.13152           | -0.18864          |
| 2.8                        | 0.9034                        | 0.9974          | 3.8772  | 0.0218 | 0.09664           | -0.16034          |
| 2.9                        | 0.9187                        | 0.9984          | 4.0768  | 0.0146 | 0.08128           | -0.14584          |
| (c <sub>1W</sub> = 0.001)  |                               |                 |         |        |                   |                   |
| 0                          | 0                             | 0               | -0.0017 | 1.3196 | 10.0              | -3.8119           |
| 0.2                        | 0.0772                        | 0.1320          | 0.0243  | 1.3178 | 9.2278            | -3.8107           |
| 0.4                        | 0.1543                        | 0.2632          | 0.1025  | 1.3023 | 8.4573            | -3.8007           |
| 0.6                        | 0.2309                        | 0.3916          | 0.2322  | 1.2605 | 7.6913            | -3.7731           |
| 0.8                        | 0.3066                        | 0.5141          | 0.4118  | 1.1828 | 6.9341            | -3.7199           |
| 1.0                        | 0.3808                        | 0.6269          | 0.6384  | 1.0657 | 6.1916            | -3.6345           |
| 1.2                        | 0.4529                        | 0.7261          | 0.9074  | 0.9136 | 5.4706            | -3.5120           |
| 1.4                        | 0.5221                        | 0.8089          | 1.2128  | 0.7395 | 4.7790            | -3.3505           |
| 1.6                        | 0.5876                        | 0.8739          | 1.5480  | 0.5613 | 4.1243            | -3.1514           |
| 1.8                        | 0.6486                        | 0.9216          | 1.9058  | 0.3975 | 3.5136            | -2.9188           |
| 2.0                        | 0.7047                        | 0.9543          | 2.2798  | 0.2616 | 2.9529            | -2.6600           |
| 2.2                        | 0.7553                        | 0.9751          | 2.6646  | 0.1596 | 2.4465            | -2.2422           |
| 2.4                        | 0.8003                        | 0.9873          | 3.0562  | 0.0900 | 1.9967            | -2.0999           |
| 2.6                        | 0.8396                        | 0.9940          | 3.4516  | 0.0470 | 1.6041            | -1.8181           |
| 2.8                        | 0.8733                        | 0.9978          | 3.8492  | 0.0226 | 1.2672            | -1.5467           |
| 2.9                        | 0.8881                        | 0.9983          | 4.0484  | 0.0152 | 1.1188            | -1.4172           |

Table 3 (Contd.)

| $\eta$                   | $\frac{c_{1w} - c_1}{c_{1w}}$ | $\frac{U}{U_e}$ | $y_0$   | $y_2$  | $y_3$    | $y_4 \times 10^2$ |
|--------------------------|-------------------------------|-----------------|---------|--------|----------|-------------------|
| (c <sub>1w</sub> = 0.05) |                               |                 |         |        |          |                   |
| 0                        | 0                             | 0               | -0.0631 | 1.0903 | 0.05     | -1.3245           |
| 0.2                      | 0.0874                        | 0.1134          | -0.0492 | 1.1036 | 0.045632 | -1.3301           |
| 0.4                      | 0.1719                        | 0.2273          | -0.0062 | 1.1105 | 0.041404 | -1.3330           |
| 0.6                      | 0.2535                        | 0.3410          | 0.0681  | 1.1041 | 0.037324 | -1.3304           |
| 0.8                      | 0.3319                        | 0.4526          | 0.1757  | 1.0776 | 0.033406 | -1.3202           |
| 1.0                      | 0.4066                        | 0.5599          | 0.3180  | 1.0253 | 0.029668 | -1.3001           |
| 1.2                      | 0.4774                        | 0.6602          | 0.4958  | 0.9446 | 0.026129 | -1.2686           |
| 1.4                      | 0.5438                        | 0.7506          | 0.7081  | 0.8367 | 0.022810 | -1.2247           |
| 1.6                      | 0.6054                        | 0.8288          | 0.9540  | 0.7077 | 0.019727 | -1.1684           |
| 1.8                      | 0.6621                        | 0.8932          | 1.2307  | 0.5680 | 0.016896 | -1.1003           |
| 2.0                      | 0.7135                        | 0.9433          | 1.5346  | 0.4302 | 0.014324 | -1.0220           |
| 2.2                      | 0.7596                        | 0.9832          | 1.8617  | 0.3058 | 0.012018 | -0.9357           |
| 2.4                      | 0.8005                        | 1.0055          | 2.2079  | 0.2033 | 0.009973 | -0.8440           |
| 2.6                      | 0.8363                        | 1.0218          | 2.5695  | 0.1259 | 0.008184 | -0.7497           |
| 2.8                      | 0.8672                        | 1.0316          | 2.9431  | 0.0724 | 0.006638 | -0.6558           |
| 2.9                      | 0.8810                        | 1.0347          | 3.1336  | 0.0534 | 0.005952 | -0.6097           |
| (c <sub>1w</sub> = 0.2)  |                               |                 |         |        |          |                   |
| 0                        | 0                             | 0               | -0.1462 | 0.6429 | 0.2000   | -2.5853           |
| 0.2                      | 0.0934                        | 0.0792          | -0.1418 | 0.6658 | 0.1813   | -2.6451           |
| 0.4                      | 0.1823                        | 0.1594          | -0.1274 | 0.6876 | 0.1635   | -2.6983           |
| 0.6                      | 0.2665                        | 0.2404          | -0.1014 | 0.7063 | 0.1467   | -2.7413           |
| 0.8                      | 0.3457                        | 0.3219          | -0.0622 | 0.7198 | 0.1309   | -2.7704           |
| 1.0                      | 0.4197                        | 0.4030          | -0.0077 | 0.7257 | 0.1160   | -2.7824           |
| 1.2                      | 0.4885                        | 0.4829          | 0.0641  | 0.7214 | 0.1023   | -2.7744           |
| 1.4                      | 0.5518                        | 0.5606          | 0.1552  | 0.7047 | 0.0896   | -2.7444           |
| 1.6                      | 0.6098                        | 0.6346          | 0.2676  | 0.6738 | 0.0780   | -2.6909           |
| 1.8                      | 0.6624                        | 0.7037          | 0.4030  | 0.6278 | 0.0675   | -2.6137           |
| 2.0                      | 0.7098                        | 0.7665          | 0.5625  | 0.5676 | 0.0580   | -2.5133           |
| 2.2                      | 0.7521                        | 0.8218          | 0.7469  | 0.4957 | 0.0496   | -2.3915           |
| 2.4                      | 0.7897                        | 0.8689          | 0.9562  | 0.4160 | 0.0421   | -2.2508           |
| 2.6                      | 0.8228                        | 0.9073          | 1.1899  | 0.3341 | 0.0354   | -2.0944           |
| 2.8                      | 0.8517                        | 0.9373          | 1.4467  | 0.2554 | 0.0296   | -1.9264           |
| 2.9                      | 0.8648                        | 0.9494          | 1.5832  | 0.2190 | 0.0270   | -1.8392           |

Table 3 (Contd.)

| $\eta$                  | $\frac{C_{1W} - C_1}{C_{1W}}$ | $\frac{U}{U_e}$ | $y_0$   | $y_2$  | $y_3$  | $y_4 \times 10^2$ |
|-------------------------|-------------------------------|-----------------|---------|--------|--------|-------------------|
| (C <sub>1W</sub> = 0.4) |                               |                 |         |        |        |                   |
| 0                       | 0                             | 0               | -0.2044 | 0.4116 | 0.4000 | -2.7106           |
| 0.2                     | 0.0860                        | 0.0618          | -0.2024 | 0.4368 | 0.3656 | -2.8654           |
| 0.4                     | 0.1698                        | 0.1254          | -0.1958 | 0.4622 | 0.3321 | -3.0134           |
| 0.6                     | 0.2509                        | 0.1906          | -0.1838 | 0.4870 | 0.2996 | -3.1498           |
| 0.8                     | 0.3285                        | 0.2571          | -0.1652 | 0.5103 | 0.2686 | -3.2700           |
| 1.0                     | 0.4020                        | 0.3244          | -0.1388 | 0.5308 | 0.2392 | -3.3693           |
| 1.2                     | 0.4709                        | 0.3922          | -0.1033 | 0.5473 | 0.2116 | -3.4436           |
| 1.4                     | 0.5347                        | 0.4597          | -0.0570 | 0.5583 | 0.1861 | -3.4896           |
| 1.6                     | 0.5934                        | 0.5263          | 0.0020  | 0.5621 | 0.1626 | -3.5046           |
| 1.8                     | 0.6467                        | 0.5913          | 0.0754  | 0.5573 | 0.1413 | -3.4873           |
| 2.0                     | 0.6948                        | 0.6536          | 0.1653  | 0.5425 | 0.1221 | -3.4370           |
| 2.2                     | 0.7378                        | 0.7124          | 0.2735  | 0.5169 | 0.1049 | -3.3546           |
| 2.4                     | 0.7760                        | 0.7668          | 0.4018  | 0.4805 | 0.0896 | -3.2418           |
| 2.6                     | 0.8096                        | 0.8159          | 0.5516  | 0.4340 | 0.0762 | -3.1012           |
| 2.8                     | 0.8390                        | 0.8590          | 0.7239  | 0.3795 | 0.0644 | -2.9364           |
| 2.9                     | 0.8522                        | 0.8781          | 0.8186  | 0.3500 | 0.0591 | -2.8462           |
| (C <sub>1W</sub> = 0.6) |                               |                 |         |        |        |                   |
| 0                       | 0                             | 0               | -0.2468 | 0.2654 | 0.6000 | -2.1815           |
| 0.2                     | 0.0674                        | 0.0467          | -0.2457 | 0.2884 | 0.5596 | -2.4018           |
| 0.4                     | 0.1367                        | 0.0960          | -0.2423 | 0.3125 | 0.5180 | -2.6262           |
| 0.6                     | 0.2070                        | 0.1478          | -0.2359 | 0.3373 | 0.4758 | -2.8493           |
| 0.8                     | 0.2774                        | 0.2018          | -0.2261 | 0.3623 | 0.4336 | -3.0652           |
| 1.0                     | 0.3467                        | 0.2578          | -0.2121 | 0.3869 | 0.3920 | -3.2670           |
| 1.2                     | 0.4141                        | 0.3154          | -0.1930 | 0.4102 | 0.3516 | -3.4484           |
| 1.4                     | 0.4785                        | 0.3740          | -0.1678 | 0.4315 | 0.2769 | -3.7261           |
| 1.6                     | 0.5393                        | 0.4334          | -0.1353 | 0.4495 | 0.2769 | -3.7261           |
| 1.8                     | 0.5959                        | 0.4928          | -0.0940 | 0.4633 | 0.2424 | -3.8130           |
| 2.0                     | 0.6479                        | 0.5516          | -0.0425 | 0.4714 | 0.2113 | -3.8609           |
| 2.2                     | 0.6950                        | 0.6092          | 0.0211  | 0.4728 | 0.1830 | -3.8686           |
| 2.4                     | 0.7374                        | 0.6648          | 0.0986  | 0.4663 | 0.1576 | -3.8361           |
| 2.6                     | 0.7751                        | 0.7177          | 0.1918  | 0.4511 | 0.1350 | -3.7647           |
| 2.8                     | 0.8083                        | 0.7671          | 0.3026  | 0.4289 | 0.1150 | -3.6570           |
| 2.9                     | 0.8234                        | 0.7902          | 0.3652  | 0.4115 | 0.1060 | -3.5905           |



Table 3 (Contd.)

| $\eta$                  | $\frac{c_{1W} - c_1}{c_{1W}}$ | $\frac{U}{U_e}$ | $y_0$   | $y_2$  | $y_3$  | $y_4 \times 10^2$ |
|-------------------------|-------------------------------|-----------------|---------|--------|--------|-------------------|
| (c <sub>1W</sub> = 0.8) |                               |                 |         |        |        |                   |
| 0                       | 0                             | 0               | -0.2871 | 0.1480 | 0.8000 | -1.2689           |
| 0.2                     | 0.0396                        | 0.0296          | -0.2866 | 0.1650 | 0.7683 | -1.4698           |
| 0.4                     | 0.0836                        | 0.0621          | -0.2859 | 0.1836 | 0.7332 | -1.6920           |
| 0.6                     | 0.1317                        | 0.0974          | -0.2819 | 0.2036 | 0.6946 | -1.9334           |
| 0.8                     | 0.1837                        | 0.1357          | -0.2773 | 0.2250 | 0.6530 | -2.1906           |
| 1.0                     | 0.2390                        | 0.1769          | -0.2708 | 0.2476 | 0.6088 | -2.4585           |
| 1.2                     | 0.2967                        | 0.2208          | -0.2618 | 0.2710 | 0.5626 | -2.7306           |
| 1.4                     | 0.3560                        | 0.2672          | -0.2500 | 0.2948 | 0.5152 | -2.9990           |
| 1.6                     | 0.4157                        | 0.3159          | -0.2347 | 0.3184 | 0.4675 | -3.2550           |
| 1.8                     | 0.4746                        | 0.3664          | -0.2152 | 0.3412 | 0.4203 | -3.4899           |
| 2.0                     | 0.5318                        | 0.4183          | -0.1905 | 0.3623 | 0.3746 | -3.6954           |
| 2.2                     | 0.5862                        | 0.4711          | -0.1595 | 0.3808 | 0.3310 | -3.8646           |
| 2.4                     | 0.6372                        | 0.5242          | -0.1211 | 0.3958 | 0.2902 | -3.9920           |
| 2.6                     | 0.6843                        | 0.5770          | -0.0738 | 0.4062 | 0.2526 | -4.0740           |
| 2.8                     | 0.7270                        | 0.6290          | -0.0159 | 0.4110 | 0.2184 | -4.1089           |
| 2.9                     | 0.7467                        | 0.6544          | 0.0176  | 0.4109 | 0.2026 | -4.1088           |

Table 4

SOLUTION OF THE LAMINAR BINARY BOUNDARY  
LAYER WITH IODINE INJECTION

| $\eta$                     | $\frac{c_{1W} - c_1}{c_{1W}}$ | $\frac{U}{U_e}$ | $y_0$   | $y_2$  | $y_3 \times 10^4$ | $y_4 \times 10^4$ |
|----------------------------|-------------------------------|-----------------|---------|--------|-------------------|-------------------|
| (c <sub>1v</sub> = 0.0001) |                               |                 |         |        |                   |                   |
| 0                          | 0                             | 0               | 0       | 1.3282 | 1.0               | -0.78718          |
| 0.2                        | 0.1573                        | 0.1328          | 0.0265  | 1.3258 | 0.84269           | -0.78492          |
| 0.4                        | 0.3130                        | 0.2647          | 0.1061  | 1.3095 | 0.68695           | -0.76925          |
| 0.6                        | 0.4633                        | 0.3938          | 0.2379  | 1.2664 | 0.53670           | -0.72833          |
| 0.8                        | 0.6022                        | 0.5167          | 0.4203  | 1.1867 | 0.39781           | -0.65511          |
| 1.0                        | 0.7232                        | 0.6298          | 0.6500  | 1.0670 | 0.27675           | -0.55092          |
| 1.2                        | 0.8212                        | 0.7290          | 0.9223  | 0.9124 | 0.17879           | -0.42683          |
| 1.4                        | 0.8938                        | 0.8115          | 1.2310  | 0.7360 | 0.10616           | -0.30076          |
| 1.6                        | 0.9426                        | 0.8761          | 1.5691  | 0.5565 | 0.05740           | -0.19068          |
| 1.8                        | 0.9720                        | 0.9233          | 1.9295  | 0.3924 | 0.02804           | -0.10786          |
| 2.0                        | 0.9877                        | 0.9555          | 2.3057  | 0.2569 | 0.01230           | -0.05410          |
| 2.2                        | 0.9952                        | 0.9759          | 2.6923  | 0.1559 | 0.00482           | -0.02396          |
| 2.4                        | 0.9983                        | 0.9878          | 3.0853  | 0.0875 | 0.00168           | -0.00934          |
| 2.6                        | 0.9995                        | 0.9942          | 3.4818  | 0.0454 | 0.00053           | -0.00320          |
| 2.8                        | 0.9998                        | 0.9975          | 3.8803  | 0.0217 | 0.00015           | -0.00096          |
| 2.9                        | 0.9999                        | 0.9984          | 4.0799  | 0.0146 | 0.00008           | -0.00050          |
| (c <sub>1v</sub> = 0.001)  |                               |                 |         |        |                   |                   |
| 0                          | 0                             | 0               | -0.0005 | 1.3278 | 10.0              | -7.8708           |
| 0.2                        | 0.1572                        | 0.1327          | 0.0261  | 1.3256 | 8.4281            | -7.8494           |
| 0.4                        | 0.3128                        | 0.2646          | 0.1057  | 1.3094 | 6.8716            | -7.6937           |
| 0.6                        | 0.4630                        | 0.3937          | 0.2375  | 1.2663 | 5.3694            | -7.2856           |
| 0.8                        | 0.6020                        | 0.5167          | 0.4200  | 1.1867 | 3.9805            | -6.5542           |
| 1.0                        | 0.7230                        | 0.6297          | 0.6497  | 1.0671 | 2.7696            | -5.5126           |
| 1.2                        | 0.8210                        | 0.7289          | 0.9220  | 0.9125 | 1.7894            | -4.2716           |
| 1.4                        | 0.8937                        | 0.8114          | 1.2307  | 0.7362 | 1.0626            | -3.0103           |
| 1.6                        | 0.9425                        | 0.8760          | 1.5688  | 0.5567 | 0.5746            | -1.9088           |
| 1.8                        | 0.9719                        | 0.9233          | 1.9292  | 0.3925 | 0.2808            | -1.0798           |
| 2.0                        | 0.9877                        | 0.9555          | 2.3054  | 0.2570 | 0.1232            | -0.5417           |
| 2.2                        | 0.9952                        | 0.9759          | 2.6920  | 0.1560 | 0.0483            | -0.2399           |
| 2.4                        | 0.9983                        | 0.9878          | 3.0850  | 0.0875 | 0.0169            | -0.0936           |
| 2.6                        | 0.9995                        | 0.9942          | 3.4816  | 0.0454 | 0.0053            | -0.0321           |
| 2.8                        | 0.9998                        | 0.9962          | 3.8806  | 0.0217 | 0.0028            | -0.0097           |
| 2.9                        | 0.9999                        | 0.9989          | 4.0796  | 0.0146 | 0.0008            | -0.0050           |

Table 4 (Contd.)

| $\eta$                    | $\frac{c_{1W} - c_1}{c_{1W}}$ | $\frac{U}{U_e}$ | $y_0$   | $y_2$  | $y_3 \times 10^3$ | $y_4 \times 10^3$ |
|---------------------------|-------------------------------|-----------------|---------|--------|-------------------|-------------------|
| (c <sub>1W</sub> = 0.005) |                               |                 |         |        |                   |                   |
| 0                         | 0                             | 0               | -0.0024 | 1.3260 | 5.0               | -3.9332           |
| 0.2                       | 0.1566                        | 0.1325          | 0.0242  | 1.3243 | 4.2168            | -3.9250           |
| 0.4                       | 0.3119                        | 0.2643          | 0.1039  | 1.3086 | 3.4403            | -3.8496           |
| 0.6                       | 0.4620                        | 0.3933          | 0.2359  | 1.2660 | 2.6901            | -3.6479           |
| 0.8                       | 0.6009                        | 0.5163          | 0.4185  | 1.1868 | 1.9956            | -3.2839           |
| 1.0                       | 0.7221                        | 0.6293          | 0.6483  | 1.6675 | 1.3894            | -2.7638           |
| 1.2                       | 0.8203                        | 0.7286          | 0.9207  | 0.9131 | 0.8983            | -2.1430           |
| 1.4                       | 0.8932                        | 0.8112          | 1.2294  | 0.7369 | 0.5337            | -1.5111           |
| 1.6                       | 0.9422                        | 0.8758          | 1.5675  | 0.5573 | 0.2888            | -0.9587           |
| 1.8                       | 0.9718                        | 0.9232          | 1.9280  | 0.3920 | 0.1412            | -0.5426           |
| 2.0                       | 0.9876                        | 0.9554          | 2.3042  | 0.2575 | 0.0620            | -0.2723           |
| 2.2                       | 0.9951                        | 0.9758          | 2.6908  | 0.1563 | 0.0243            | -0.1207           |
| 2.4                       | 0.9983                        | 0.9878          | 3.0837  | 0.0877 | 0.0085            | -0.0471           |
| 2.6                       | 0.9995                        | 0.9942          | 3.4802  | 0.0455 | 0.0026            | -0.0162           |
| 2.8                       | 0.9998                        | 0.9975          | 3.8786  | 0.0218 | 0.0008            | -0.0049           |
| 2.9                       | 0.9999                        | 0.9989          | 4.0782  | 0.0146 | 0.0004            | -0.0025           |
| (c <sub>1W</sub> = 0.01)  |                               |                 |         |        |                   |                   |
| 0                         | 0                             | 0               | -0.0049 | 1.3238 | 10.0              | -7.8608           |
| 0.2                       | 0.1559                        | 0.1323          | 0.0218  | 1.3227 | 8.4406            | -7.8506           |
| 0.4                       | 0.3108                        | 0.2640          | 0.1016  | 1.3076 | 6.8922            | -7.7062           |
| 0.6                       | 0.4606                        | 0.3928          | 0.2339  | 1.2656 | 5.3938            | -7.3084           |
| 0.8                       | 0.5995                        | 0.5158          | 0.4166  | 1.1869 | 4.0047            | -6.5848           |
| 1.0                       | 0.7209                        | 0.6288          | 0.6466  | 1.1068 | 2.7906            | -5.5467           |
| 1.2                       | 0.8194                        | 0.7282          | 0.9191  | 0.9138 | 1.8054            | -4.3042           |
| 1.4                       | 0.8926                        | 0.8108          | 1.2278  | 0.7377 | 1.0735            | -3.0373           |
| 1.6                       | 0.9419                        | 0.8756          | 1.5660  | 0.5581 | 0.5812            | -1.9282           |
| 1.8                       | 0.9716                        | 0.9230          | 1.9264  | 0.3937 | 0.2842            | -1.0920           |
| 2.0                       | 0.9875                        | 0.9553          | 2.3025  | 0.2580 | 0.1248            | -0.5484           |
| 2.2                       | 0.9951                        | 0.9757          | 2.6891  | 0.1566 | 0.0490            | -0.2431           |
| 2.4                       | 0.9983                        | 0.9877          | 3.0820  | 0.0880 | 0.0171            | -0.0949           |
| 2.6                       | 0.9995                        | 0.9942          | 3.4786  | 0.0456 | 0.0054            | -0.0326           |
| 2.8                       | 0.9998                        | 0.9975          | 3.8770  | 0.0219 | 0.0015            | -0.0098           |
| 2.9                       | 0.9999                        | 0.9998          | 4.0766  | 0.0147 | 0.0008            | -0.0051           |

Table 4 (Contd.)

| $\eta$                  | $\frac{C_{1W} - C_1}{C_{1W}}$ | $\frac{U}{U_e}$ | $y_0$   | $y_2$  | $y_3$   | $y_4$   |
|-------------------------|-------------------------------|-----------------|---------|--------|---------|---------|
| (C <sub>1W</sub> = 0.2) |                               |                 |         |        |         |         |
| 0                       | 0                             | 0               | -0.1167 | 1.2226 | 0.2000  | -0.1521 |
| 0.2                     | 0.1288                        | 0.1224          | -0.0876 | 1.2488 | 0.1742  | -0.1566 |
| 0.4                     | 0.2644                        | 0.2471          | -0.0016 | 1.2610 | 0.1471  | -0.1588 |
| 0.6                     | 0.4036                        | 0.3718          | 0.1387  | 1.2451 | 0.1193  | -0.1559 |
| 0.8                     | 0.5408                        | 0.4932          | 0.3296  | 1.1894 | 0.0918  | -0.1456 |
| 1.0                     | 0.6683                        | 0.6070          | 0.5664  | 1.0886 | 0.0663  | -0.1272 |
| 1.2                     | 0.7778                        | 0.7087          | 0.8435  | 0.9464 | 0.0444  | -0.1023 |
| 1.4                     | 0.8636                        | 0.7948          | 1.1545  | 0.7757 | 0.0273  | -0.0747 |
| 1.6                     | 0.9239                        | 0.8633          | 1.4929  | 0.5957 | 0.0152  | -0.0489 |
| 1.8                     | 0.9617                        | 0.9142          | 1.8526  | 0.4265 | 0.0076  | -0.0285 |
| 2.0                     | 0.9827                        | 0.9495          | 2.2275  | 0.2837 | 0.0034  | -0.0147 |
| 2.2                     | 0.9930                        | 0.9721          | 2.6130  | 0.1749 | 0.0014  | -0.0067 |
| 2.4                     | 0.9974                        | 0.9856          | 3.0051  | 0.0997 | 0.0005  | -0.0027 |
| 2.6                     | 0.9991                        | 0.9930          | 3.4011  | 0.0526 | 0.0002  | -0.0009 |
| 2.8                     | 0.9996                        | 0.9968          | 3.7992  | 0.0256 | 0.0000  | -0.0003 |
| 2.9                     | 0.9998                        | 0.9978          | 3.9987  | 0.0173 | 0.0000  | -0.0002 |
| (C <sub>1W</sub> = 0.4) |                               |                 |         |        |         |         |
| 0                       | 0                             | 0               | -0.2988 | 1.0581 | 0.4000  | -0.2922 |
| 0.2                     | 0.1000                        | 0.1071          | -0.2671 | 1.1198 | 0.3600  | -0.3110 |
| 0.4                     | 0.2120                        | 0.2199          | -0.1742 | 1.1706 | 0.3152  | -0.3274 |
| 0.6                     | 0.3354                        | 0.3368          | -0.0246 | 1.1948 | 0.2658  | -0.3357 |
| 0.8                     | 0.4670                        | 0.4544          | 0.1761  | 1.1780 | 0.2132  | -0.3295 |
| 1.0                     | 0.5998                        | 0.5682          | 0.4212  | 1.1111 | 0.1601  | -0.3039 |
| 1.2                     | 0.7242                        | 0.6731          | 0.7034  | 0.9942 | 0.1103  | -0.2587 |
| 1.4                     | 0.8302                        | 0.7646          | 1.0158  | 0.8382 | 0.0679  | -0.1999 |
| 1.6                     | 0.9107                        | 0.8394          | 1.3523  | 0.6621 | 0.0357  | -0.1383 |
| 1.8                     | 0.9648                        | 0.8968          | 1.7077  | 0.4879 | 0.0141  | -0.0849 |
| 2.0                     | 0.9967                        | 0.9376          | 2.0774  | 0.3343 | 0.0013  | -0.0460 |
| 2.2                     | 1.0131                        | 0.9647          | 2.4574  | 0.2124 | -0.0052 | -0.0219 |
| 2.4                     | 1.0206                        | 0.9813          | 2.8445  | 0.1250 | -0.0082 | -0.0092 |
| 2.6                     | 1.0235                        | 0.9907          | 3.2360  | 0.0680 | -0.0094 | -0.0034 |
| 2.8                     | 1.0246                        | 0.9957          | 3.6301  | 0.0342 | -0.0098 | -0.0011 |
| 2.9                     | 1.0248                        | 0.9971          | 3.8276  | 0.0236 | -0.0099 | -0.0006 |

Table 4 (Contd.)

| $\eta$                  | $\frac{c_{1W} - c_1}{c_{1W}}$ | $\frac{U}{U_e}$ | $y_0$   | $y_2$  | $y_3$   | $y_4$   |
|-------------------------|-------------------------------|-----------------|---------|--------|---------|---------|
| (c <sub>1W</sub> = 0.6) |                               |                 |         |        |         |         |
| 0                       | 0                             | 0               | -0.6018 | 0.8110 | 0.6000  | -0.3924 |
| 0.2                     | 0.0666                        | 0.0846          | -0.5680 | 0.9108 | 0.5600  | -0.4308 |
| 0.4                     | 0.1455                        | 0.1790          | -0.4685 | 1.0095 | 0.5127  | -0.4711 |
| 0.6                     | 0.2386                        | 0.2823          | -0.3076 | 1.0904 | 0.4568  | -0.5067 |
| 0.8                     | 0.3464                        | 0.3919          | -0.0919 | 1.1350 | 0.3921  | -0.5279 |
| 1.0                     | 0.4669                        | 0.5036          | 0.1696  | 1.1271 | 0.3198  | -0.5236 |
| 1.2                     | 0.5937                        | 0.6120          | 0.4670  | 1.0590 | 0.2438  | -0.4844 |
| 1.4                     | 0.7160                        | 0.7110          | 0.7912  | 0.9354 | 0.1704  | -0.4096 |
| 1.6                     | 0.8216                        | 0.7960          | 1.1350  | 0.7728 | 0.1070  | -0.3109 |
| 1.8                     | 0.9017                        | 0.8642          | 1.4935  | 0.5950 | 0.0590  | -0.2089 |
| 2.0                     | 0.9543                        | 0.9149          | 1.8636  | 0.4257 | 0.0274  | -0.1233 |
| 2.2                     | 0.9843                        | 0.9500          | 2.2429  | 0.2825 | 0.0094  | -0.0638 |
| 2.4                     | 0.9990                        | 0.9725          | 2.6293  | 0.1736 | 0.0006  | -0.0289 |
| 2.6                     | 1.0054                        | 0.9859          | 3.0206  | 0.0986 | -0.0032 | -0.0115 |
| 2.8                     | 1.0078                        | 0.9932          | 3.4152  | 0.0518 | -0.0047 | -0.0040 |
| 2.9                     | 1.0083                        | 0.9954          | 3.6132  | 0.0365 | -0.0050 | -0.0022 |
| (c <sub>1W</sub> = 0.8) |                               |                 |         |        |         |         |
| 0                       | 0                             | 0               | -1.2249 | 0.3702 | 0.8000  | -0.3993 |
| 0.2                     | 0.0321                        | 0.0424          | -1.1983 | 0.4733 | 0.7743  | -0.4502 |
| 0.4                     | 0.0713                        | 0.0960          | -1.1154 | 0.5977 | 0.7429  | -0.5096 |
| 0.6                     | 0.1200                        | 0.1624          | -0.9730 | 0.7368 | 0.7040  | -0.5760 |
| 0.8                     | 0.1807                        | 0.2424          | -0.7707 | 0.8766 | 0.6554  | -0.6451 |
| 1.0                     | 0.2562                        | 0.3350          | -0.5127 | 0.9957 | 0.5950  | -0.7082 |
| 1.2                     | 0.3486                        | 0.4369          | -0.2076 | 1.0695 | 0.5211  | -0.7515 |
| 1.4                     | 0.4578                        | 0.5429          | 0.1321  | 1.0779 | 0.4337  | -0.7566 |
| 1.6                     | 0.5790                        | 0.6463          | 0.4936  | 1.0139 | 0.3368  | -0.7071 |
| 1.8                     | 0.7013                        | 0.7404          | 0.8662  | 0.8867 | 0.2389  | -0.5990 |
| 2.0                     | 0.8101                        | 0.8202          | 1.2437  | 0.7196 | 0.1519  | -0.4502 |
| 2.2                     | 0.8933                        | 0.8828          | 1.6242  | 0.5411 | 0.0854  | -0.2959 |
| 2.4                     | 0.9474                        | 0.9283          | 2.0080  | 0.3768 | 0.0421  | -0.1690 |
| 2.6                     | 0.9773                        | 0.9590          | 2.3958  | 0.2428 | 0.0182  | -0.0840 |
| 2.8                     | 0.9915                        | 0.9780          | 2.7876  | 0.1446 | 0.0068  | -0.0364 |
| 2.9                     | 0.9951                        | 0.9843          | 2.9848  | 0.1084 | 0.0039  | -0.0228 |

Appendix B

CALCULATION OF  $H_1(c)$  AND  $K_{1,j}$

$$H_1(c) = \frac{1}{4B} \int_0^e \frac{[y_1 - y_1(c)]^2}{y_3} d\eta$$

$$K_{12} = 2.65644 \int_{\eta(a)}^e \frac{y_3^2 d\eta}{[y_3(0)]^2 [y_1 - y_1(c)]^2}$$

$$K_{10} = \frac{-2y_1(a)A}{y_1(c) [y_1(a) - y_1(c)]} + B \ln \frac{y_1(a) - y_1(c)}{y_1(c)}$$

$$K_{11}^1 = \int_0^2 \frac{\frac{y_1(a)}{2} \frac{4y_3^2}{[y_3(0)]^2} \frac{1}{[y_1 - y_1(c)]^2} - \frac{4A}{[y_1 - y_1(c)]^2} - \frac{2B}{[y_1 - y_1(c)]} d(\frac{y_1}{2})$$

$$K_{11}^2 = \frac{y_3^2}{sy_3(0)^2} \left[ \frac{y_4^1}{y_3} + \frac{y_4^2}{y_3^2} - 2 \frac{y_4 s^1}{y_3 s} + \frac{s^{1^2}}{s^2} - \frac{s''}{2s} \right]_c$$

where

$$A = \frac{y_3^2}{sy_3(0)^2} \quad B = \frac{y_3^2}{sy_3(0)^2} \quad 2 \frac{y_4}{y_3} - \frac{s'}{s} \quad c$$

$$s = \frac{\mu y_1'}{\mu(0)y_1'(0)}$$

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